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Classical Mechanics

Newton's Laws and Dynamical Systems

" What we know is a drop, what we don't know is an ocean." **Isaac Newton**

First Law: It states that a particle which is not acted upon by any external force, maintains its state of rest or state of uniform motion.

When Newton's law is applied to an object or system which consists of more than one particles, it is centre of mass of the object or system which obeys Newton's law. For example if two particles are placed on a smooth table at some separation, because of their natural gravitational attraction the two will move towards each other but centre of the system of two particles remains at rest.

Second Law: It states that rate of change of momentum of a system is equal to resultant external force on the system.

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad \text{... (2.1)}$$

This form of Newton's law is valid in all cases.

If m be mass of system and $\vec{v}$ be its velocity then $\vec{p} = m\vec{v}$

Therefore, $\vec{F} = \frac{d}{dt}(m\vec{v})$

If mass of system is constant then, $\vec{F} = m\frac{d\vec{v}}{dt}$

or $\vec{F} = m\vec{a}$ ... (2.2)

where $\vec{a} = \frac{d\vec{v}}{dt}$ is acceleration, evidently, equation (2.2) can be used only if mass is constant. If mass is variable, we use equation (2.1).

If F_x, F_y, F_z be Cartesian components of $\vec{F}$ and (x, y, z) be coordinate of the particle on which force is acting then equation (2.2) can be written in component form as

$$\left.\begin{aligned} F_x &= m\frac{d^2x}{dt^2} \\ F_y &= m\frac{d^2y}{dt^2} \\ F_z &= m\frac{d^2z}{dt^2} \end{aligned}\right\} \qquad \text{... (2.3)}$$

However, instead of Cartesian coordinates if we use some curvilinear coordinate then we do not get similar relations. For example if F_r, F_θ, F_ϕ be components of force $\vec{F}$ in spherical polar coordinate and r, θ, ϕ be spherical polar coordinates of particle then,

$$\left.\begin{aligned} F_r &\neq m\frac{d^2r}{dt^2} \\ F_\theta &\neq m\frac{d^2\theta}{dt^2} \\ F_\phi &\neq m\frac{d^2\phi}{dt^2} \end{aligned}\right\} \qquad \text{... (3.3)}$$

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Thus, we cannot write Newton's law in other coordinate system by merely replacing x, y, z by corresponding coordinates.
Due to this reason we say that Newton's law is not covariant under coordinate transformation.

Third Law: It states that to every action there is equal and opposite reaction. If $\vec{F}_{12}$ be force on an object 1 due to an object 2, and F_{21} be force on 2 due to 1 then

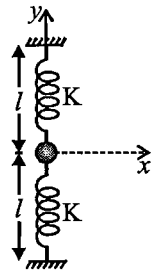
$$\vec{F}_{12} = -\vec{F}_{21}$$

Newton's third law assumes that to an action there is prompt reaction. This implies that in Newton's law it is assumed that information travels with infinite speed.

Violation of Newton's Laws: Newton's 1st and 2nd laws are strictly valid in inertial frames. In non-inertial frames Newton's laws do not hold good. In non-inertial frames we can apply Newton's law if we include pseudo forces.
There are some cases where Newton's third law also does not hold. For example magnetic force exerted by two non-parallel current carrying wires on each other are not in opposite in direction.

Example: A particle of mass m is connected to two springs of unstretched length 'l' and spring constant k as shown in figure. Calculate acceleration of particle if it is slightly displaced along X-direction.

Soln. Let the particle is displaced by a distance x along $+x$ direction as shown in figure.

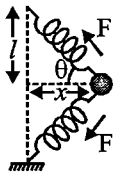


Therefore, stretched length of spring $= \sqrt{l^2 + x^2}$

elongation in springs $= \sqrt{l^2 + x^2} - l$

Restoring force on the particle due to each spring $F = k(\sqrt{l^2 + x^2} - l)$

Because of restoring force the particle moves back towards its initial position.



Equation of motion of particle $F_x = m\dfrac{d^2x}{dt^2}$

or $-2F\cos\theta = m\dfrac{d^2x}{dt^2}$ or $-2k(\sqrt{l^2 + x^2} - l).\dfrac{x}{\sqrt{l^2 + x^2}} = m\dfrac{d^2x}{dt^2}$

Therefore, $\dfrac{d^2x}{dt^2} = -\dfrac{2kx}{m}\left(1 - \dfrac{l}{\sqrt{l^2 + x^2}}\right)$ or $\dfrac{d^2x}{dt^2} = -\dfrac{2kx}{m}\left[1 - \left(1 + \dfrac{x^2}{l^2}\right)^{-1/2}\right]$

Since, $x << l, \left(1 + \dfrac{x^2}{l^2}\right)^{-1/2} \approx 1 - \dfrac{x^2}{2l^2}$

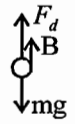
Therefore, $\dfrac{d^2x}{dt^2} = -\dfrac{2kx}{m}\left[1 - \left(1 - \dfrac{x^2}{2l^2}\right)\right]$ or $\dfrac{d^2x}{dt^2} = -\dfrac{kx^3}{ml^2}$

Example: Spherical particles of a given material of density ρ are released from rest inside a liquid medium of lower density. The viscous drag force may be approximated by the Stoke's law, i.e, $F_d = 6\pi\eta Rv$, where η is the viscosity of the medium, R the radius of a particle and v its instantaneous velocity. If $\tau(m)$ is the time taken by a particle of mass m to reach half its terminal velocity, then the ratio $\tau(8m)/\tau(m)$ is

Soln. Each particle has same density but different radii and masses. We have to calculate time of fall in terms of mass of particle therefore we will write our equations explicitly in terms of mass and we will remove radius from our equations wherever it appears.

Drag force on particles

$F_d = 6\pi\eta R\text{v}$, Mass of a particle $m = \rho.\frac{4}{3}\pi R^3$



$$\therefore R = \left(\frac{3m}{4\pi\rho}\right)^{1/3} \quad \therefore F_d = 6\pi\eta\left(\frac{3m}{4\pi\rho}\right)^{1/3} v \quad = Km^{1/3}v, \text{ where } K = 6\pi\eta\left(\frac{3m}{4\pi\rho}\right)^{1/3}$$

If B is buoyancy force, then equation of motion of a particle is

$$m\frac{dv}{dt} = mg - F_d - B \text{ where B} = \frac{4}{3}\pi R^3\sigma g = \frac{4}{3}\pi R^3\rho g.\frac{\sigma}{\rho} = mg\frac{\sigma}{\rho}, \quad \sigma = \text{density of medium}$$

$$\therefore m\frac{dv}{dt} = mg - Km^{1/3}v - mg\frac{\sigma}{\rho}, \quad m\frac{dv}{dt} = mg\left(1-\frac{\sigma}{\rho}\right) - Km^{1/3}v$$

or $$\frac{dv}{dt} = g\left(1-\frac{\sigma}{\rho}\right) - Km^{-2/3}v \quad \ldots (a)$$

when terminal velocity is reached $\frac{dv}{dt} = 0$

$$\therefore 0 = g\left(1-\frac{\sigma}{\rho}\right) - Km^{-2/3}v_t \quad \therefore v_t = \frac{g\left(1-\frac{\sigma}{\rho}\right)}{Km^{-2/3}} \quad \ldots (b)$$

Now, if τ be the time to reach half the terminal velocity then from (a)

$$\int_0^{v_t/2} \frac{dv}{g\left(1-\frac{\sigma}{\rho}\right) - Km^{-2/3}v} = \int_0^{\tau} dt = \frac{1}{Km^{-2/3}} \ln\left[\frac{g\left(1-\frac{\sigma}{\rho}\right) - \frac{Km^{-2/3}v_t}{2}}{g\left(1-\frac{\sigma}{\rho}\right)}\right]_0^{\tau}$$

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using value of v_t from (b) we get, $\tau = \frac{m^{2/3}}{K}\ln 2$ or $\tau(m) = \frac{m^{2/3}\ln 2}{K}$

$$\therefore \tau(8m)/\tau(m) = (8m)^{2/3}/m^{2/3} = 4$$

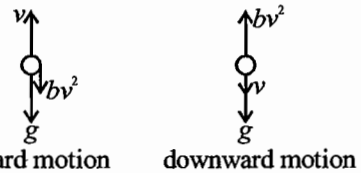
Example: A particle of unit mass is thrown vertically upward with initial speed v_0. It is acted upon by a drag force bv^2 in addition to gravity where b is constant and v is instantaneous velocity of particle. Calculate speed of the particle when it returns to the point from where it was thrown.

Soln. In presence of drag force only quantity that is common in upward and downward motion is the distance covered.

For upward motion, initial speed = v_0,

final speed = 0, let height reached = h

equation of motion $\frac{dv}{dt} = -g - bv^2$



or $\frac{dv}{dy}.\frac{dy}{dt}=-g-bv^3$, $\frac{dy}{dt}=v$ or $\frac{vdv}{g+bv^2}=-dy$

$\therefore \int_{v_0}^{0}\frac{vdv}{g+bv^2}=-\int_0^h dy$

on integration we get, $\frac{1}{2b}\ln\left(\frac{g+bv_0^2}{g}\right)=h$ ... (a)

for downward motion, initial speed = 0, final speed = v(let), height descended = h

equation of motion $\frac{dv}{dt}=g-bv^2$ or $\frac{vdv}{g-bv^2}=dy$ or $\int_0^v\frac{vdv}{g-bv^2}=\int_0^h dy$

on integration we get, $\frac{1}{2b}\ln\left(\frac{g}{g-bv^2}\right)=h$ ... (b)

Comparing (a) and (b) we get, $\frac{g+bv_0^2}{g}=\frac{g}{g-bv^2}$, $g-bv^2=\frac{g^2}{g+bv_0^2}$

or $v=\frac{v_0}{\sqrt{1+\frac{bv_0^2}{g}}}\Rightarrow v<v_0$

due to drag force the particle returns with a speed less than its initial speed. If drag force were absent $b=0$, then $v=v_0$. Therefore in absence of drag force particle returns with same speed as its initial value.

Formulae of kinematics (uniformly accelerated motions): When a constant force acts on a particle its acceleration is constant. If u_x, v_x be initial and final velocity of a particle along x-direction and a_x be its acceleration along x direction then

$$\left.\begin{aligned} v_x&=u_x+a_xt\\ v_x^2&=u_x^2+2a_xx\\ x&=u_xt+\tfrac{1}{2}a_xt^2\end{aligned}\right\}\text{if } a_x=\text{constant}$$

where x is displacement along x-direction and t is time taken. We can write similar relations for y and z direction also.

Note: If acceleration of particle is not constant then we cannot use formulae of kinematics. In that case we start with either definition of velocity or definition of acceleration i.e.

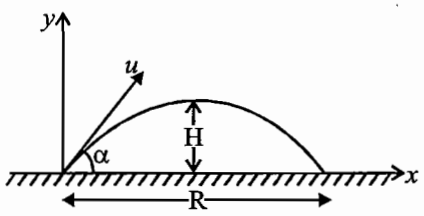
$$v_x=\frac{dx}{dt}\quad\text{or}\quad a_x=\frac{dv_x}{dt}$$

we may also have to use $a_x=\frac{F_x}{m}$ and $\frac{dv_x}{dt}=\frac{dv_x}{dx}.\frac{dx}{dt}=\frac{v_xdv_x}{dx}$

as the case may be

Projectile motion: If u be the projection speed and α be angle of projection then equation of path of projectile is,

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$



Range, $R = \dfrac{u^2\sin 2\alpha}{g}$

Maximum height, $H = \dfrac{u^2\sin^2\alpha}{2g}$

Total time of flight, $T = \dfrac{2u\sin\alpha}{g}$

Coordinate of projectile at any time $t < T$.

$$x = u\cos\alpha\, t,\ y = u\sin\alpha\, t - \frac{1}{2}gt^2$$

Note: Above results for projectile are applicable only if projectile is thrown from ground and it finally lands on ground. And projection speed is small so that projectile does not go too high. If projection speed is not small then height will be large and in that case we need to consider variation of acceleration due to gravity.

Example: A particle starts moving with acceleration $a = \alpha - \beta v$ where α and β are constants and v is its instantaneous speed. Find velocity of particle as a function of time and also find its terminal velocity.

Soln: Acceleration is variable therefore we start with definition of acceleration

$$a = \frac{dv}{dt} \text{ or } \alpha - \beta v = \frac{dv}{dt} \text{ or } \int_0^v \frac{dv}{\alpha - \beta v} = \int_0^t dt$$

$$-\frac{1}{\beta}\ln\left(\frac{\alpha - \beta v}{\alpha}\right) = t \text{ or, } 1 - \frac{\beta v}{\alpha} = e^{-\beta t} \quad \therefore v = \frac{\alpha}{\beta}(1 - e^{-\beta t})$$

as $t \to \infty$, $v = \dfrac{\alpha}{\beta}$ = constant. This is terminal velocity.

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Example: A projectile is thrown at an angle α with horizontal with initial speed u after what time the projectile's velocity will be perpendicular to its initial direction.

Soln. Suppose velocity after time t becomes perpendicular to initial velocity. Velocity after time t is

$$\vec{v}(t) = \dot{x}\hat{i} + \dot{y}\hat{j} \quad = u\cos\alpha\hat{i} + (u\sin\alpha - gt)\hat{j}$$

Initial velocity $\vec{u} = u(\cos\alpha\hat{i} + \sin\alpha\,\hat{j})$

$\vec{v}$ is $\perp$ to $\vec{u}$, therefore, $\vec{v}.\vec{u} = 0 \quad \therefore u\cos^2\alpha + u\sin^2\alpha - gt\sin\alpha = 0$

$$\therefore t = \frac{u}{g\sin\alpha}$$

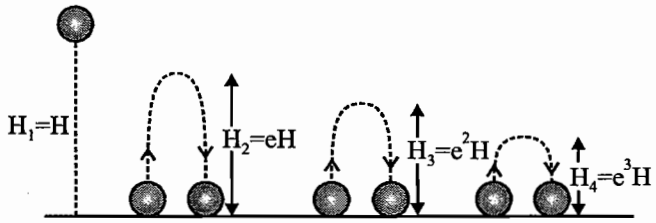
Example: A ball is dropped from a height H. It bounces back up to a height e times after hitting the ground. If $e<1$, after what time the ball will finally come to rest.

Soln. Ball moves under the effect of gravity. Therefore magnitude of acceleration of ball during upward or downward movement remains constant.

For first bounce

$$y = H,\ u_y = 0,\ a_y = g$$

$$y = u_y t + \frac{1}{2}a_y t^2$$



$$\therefore H = 0 + \frac{1}{2}gt^2 \qquad \therefore t = \sqrt{\frac{2H}{g}}$$

Therefore time of fall before first bounce $t_1 = \sqrt{\frac{2H}{g}}$

Under gravity, time for going up a certain distance is equal to time for going down. Therefore time interval between first bounce and second bounce is

$$t_2 = 2\sqrt{\frac{2H_2}{g}} = 2\sqrt{\frac{2eH}{g}}$$

Similarly $t_3 = 2\sqrt{\frac{2e^2H}{g}}$, $t_4 = 2\sqrt{\frac{2e^3H}{g}}$,.....

Therefore, total time elapsed before ball stops is

$$T = t_1 + t_2 + t_3 + ...$$

$$= \sqrt{\frac{2H}{g}}\left[1 + 2\left(e^{1/2} + e^{2/2} + e^{3/2} + ...\right)\right] = \sqrt{\frac{2H}{g}}\left[1 + 2\frac{e^{1/2}}{1 - e^{1/2}}\right] = \sqrt{\frac{2H}{g}}\left(\frac{1+\sqrt{e}}{1-\sqrt{e}}\right)$$

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Some Important Terms:

Conservative force: A force is said to be conservative if curl of the force is zero.

i.e. if $\vec{\nabla} \times \vec{F} = 0$ then, $\vec{F}$ is conservative force.

Work done by a conservative force is independent of path followed and work done along a closed path is zero. Gravitational and electrostatic forces are conservative. Friction, viscous forces are not conservative.

Potential energy: It is defined as work done by conservative force in taking the system from given configuration (position) to a reference position. Reference positions is that where the conservative force vanishes.

Potential energy can be directly obtained from relation

$U = -\int F dr$ or $U = -\int F dx$

for attractive force F is used with negative sign and for repulsive force F is used with positive sign.

Spring Force: When a spring is either stretched or compressed such that its length changes by x then restoring force or tension in the spring is $-kx$.

Spring potential energy: Particle attached to the spring is always attracted towards mean position. Therefore, $F = -kx$

$$\therefore U = -\int (-kx)dx = \frac{1}{2}kx^2$$

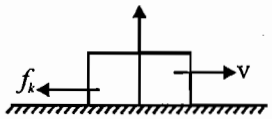
Gravitational potential energy for two particles:

We know that gravitational force between two particles of masses m_1 and m_2 at separation r is always attractive and is equal to

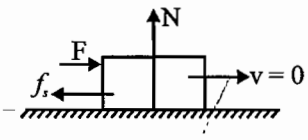
$$f = -\frac{Gm_1.m_2}{r^2} \qquad \therefore U = -\int\left(-\frac{Gm_1.m_2}{r^2}\right)dr = -\frac{Gm_1.m_2}{r}$$

Friction force: Its arise because of interaction (electromagnetic) among the particles of surfaces in contact. It is of two types

(i) Kinetic friction: It acts on objects which are sliding on a surface $f_k = \mu_k N$



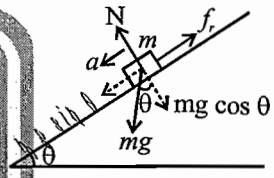
(ii) Static friction: It acts on object which are subjected to a force but not sliding on the contact surface. It is an adjustable force $f_s \le \mu_s N$



Example: A block slides down a rough inclined plane of inclination θ. Coefficient of friction between the block and the plane is μ. Calculate acceleration of block.

Soln. The block slides on the inclined plane.

Therefore, to get acceleration of block we resolve all forces parallel and perpendicular to the plane.



Therefore, for equilibrium perpendicular to the plane

$N = mg\cos\theta$ ... (i)

equation of motion parallel to plane, $mg\sin\theta - f_r = ma$ ... (ii)

and, $f_r = \mu N = \mu mg\cos\theta$

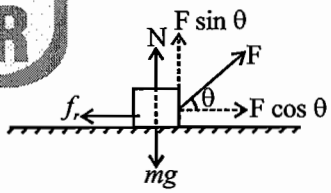
from (ii), $mg\sin\theta - \mu mg\cos\theta = ma \quad \therefore a = g(\sin\theta - \mu\cos\theta)$

Example: A block of mass 'm' is pulled with a force F at an angle θ with horizontal. The block does not move on the surface. Calculate force of friction on the block and value of coefficient of friction.

Soln. The block is stationary, therefore horizontal components of F must be balanced by friction force.

$\therefore f_r = F\cos\theta$

for vertical equilibrium, $N + F\sin\theta = mg$



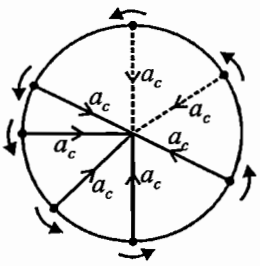
or, $N = mg - F\sin\theta$

if μ be coefficient of static friction then,

$f_r \le \mu N$

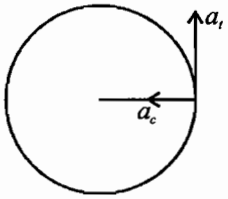
or $F\cos\theta \le \mu(mg - F\sin\theta) \quad \therefore \mu \ge \dfrac{F\cos\theta}{mg - F\sin\theta}$

Centripetal acceleration: When a particle moves in a circular path, due to continuous change in direction of velocity of the particle, particle always have an acceleration towards centre of circle.



This is called centripetal acceleration. Its value is $a_c = \dfrac{v^2}{r}$

If the particle's velocity is non-uniform then in addition to centripetal acceleration, the particle also has a tangential acceleration which is equal to $a_t = \dfrac{dv}{dt}$



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Centripetal force: It is not a new kind of force. When an object moves in a circular path the net components of known forces towards centre of circle is called centripetal force.

It turns out that if a particle of mass m is moving in a circle of radius r. Then at any instant when its speed is v, net force towards centre $= ma_c = \frac{mv^2}{r}$.

Example: A point mass m is tied to a light string of length 'l'. It is whirled in a vertical circle. Its speeds at lowest point, highest point and at a point where string makes 60° with upward vertical are $\sqrt{6gl}, \sqrt{2gl}$ and $\sqrt{3gl}$ respectively. Calculate tension in string at these points.

Soln. The point mass is moving in a circle, therefore at each point of circle net force towards centre must be equal to $\frac{mv^2}{r}$.

At lowest point: Net force towards centre = $T - mg$

$\therefore \; T - mg = \frac{mv^2}{l}$ or $T = mg + \frac{m6gl}{l} = 7mg$

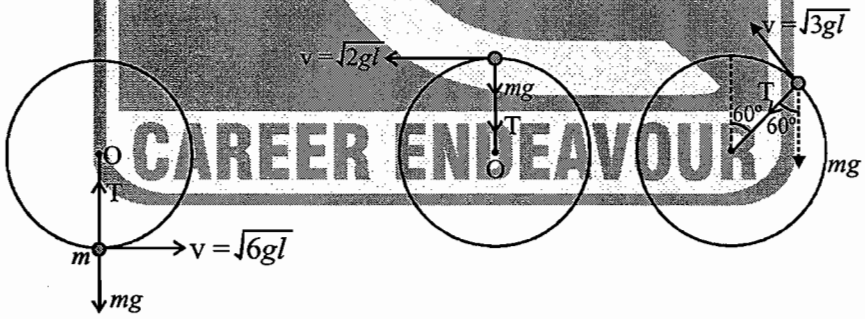
At highest point: Net force towards centre = $T + mg$

$\therefore T + mg = \frac{mv^2}{l}$ or $T = \frac{m2gl}{l} - mg = mg$

At 60° with upward vertical: Net force towards centre = $T + mg\cos 60°$

$\therefore T + mg\cos 60° = \frac{mv^2}{l}$

$T = \frac{m.3gl}{l} - \frac{mg}{2} = \frac{5mg}{2}$



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Work: It is defined as $W = \int \vec{F}.\vec{dl}$

where $\vec{dl}$ is infinitesimal displacement of the object on which force is acting.

Power: It is defined as dot product of force and instantaneous velocity $P = \vec{F}.\vec{v}$

Average power is defined as total work done divided by total time $\langle P \rangle = \frac{W}{t}$

Kinetic Energy: For a particle of mass 'm' and speed v, kinetic energy is defined as $K = \frac{1}{2}mv^2$

In terms of momentum it is written as $K = \frac{p^2}{2m}$.

Work-energy theorem: It states that total work done by all forces is equal to change in kinetic energy of system.

$$W = K_{final} - K_{initial}$$

Conservation of mechanical energy: If only conservative forces are acting on a system then total mechanical energy of system remains conserved. Therefore

$$E_{initial} = E_{final}$$

or $\quad K_i + U_i = K_f + U_f \;\Rightarrow\; U_i - U_f = K_f - K_i$

Therefore, loss in potential energy = gain in kinetic energy (or vice versa).

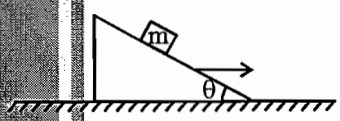
Example: A particle moves from point (1, –1, 2) to (2, 3, 5) when a force $\vec{F} = 3\hat{i} + 5\hat{j} + 4\hat{k}$ acts on it. Calculate work done by the force.

Soln. The given force is constant. Therefore it is a conservative force. Therefore work done is independent of path followed.

$$W = \int \vec{F}.d\vec{l} = \int F_x.dx + \int F_y.dy + \int F_z.dz = \int_1^2 3dx + \int_{-1}^{3} 5dy + \int_2^5 5dz$$

$$= 3(2-1) + 5(3+1) + 5(5-2) = 38 \text{ Joule}$$

Example: A block of mass m is lying on a wedge as shown in figure. The block does not slide on the wedge while the wedge moves horizontally with uniform speed. Calculate work done by normal reaction on the block during the time wedge moves by a distance S in horizontal direction.



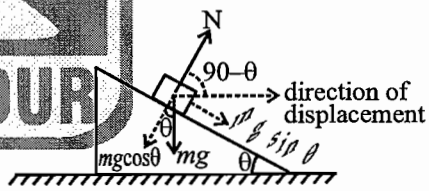
Soln. From the figure shown we find that $N = mg\cos\theta$

and angle between N and direction of displacement is $90-\theta$. Therefore work done by normal reaction is,

$$W = \int \vec{F}.d\vec{l} = \vec{N}.\int d\vec{l} = \vec{N}.\vec{S}$$

$$= N S \cos(90-\theta) = mg\cos\theta . S \sin\theta$$

$$= mg\, S \sin\theta . \cos\theta$$

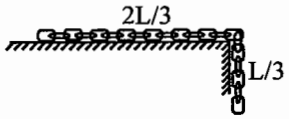


CAREER ENDEAVOUR

Example: Two third length of a thin metallic chain of mass per unit length λ is held on a smooth table and remaining hanging off from its edge. If the chain is released what will be its speed at the moment the chain completely slips off the table.

Soln. There is no dissipative force acting on the system.

Therefore total energy of the system remains conserved. Therefore we will use conservation of energy to find speed. Let us take potential energy to be zero at level of table top. Initially centre of mass of hanging part is L/6 below the table top and finally centre of mass of whole chain is L/2 below the table top. Therefore,



$$U_{initial} = -\left(\frac{\lambda L}{3}\right) g\left(\frac{L}{6}\right) = -\frac{\lambda g L^2}{18}, \quad U_{final} = -(\lambda L) g\left(\frac{L}{2}\right) = -\frac{\lambda g L^2}{2}$$

$$K_{initial} = 0, \quad K_{final} = \frac{1}{2}(\lambda L) v^2$$

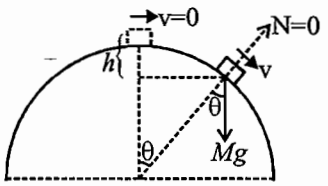
From conservation of energy $K_{initial}+U_{initial}=K_{final}+U_{final}$

$$0+\left(-\frac{\lambda gL^2}{18}\right)=\frac{1}{2}(\lambda L)v^2+\left(-\frac{\lambda gL^2}{2}\right) \Rightarrow v=\sqrt{\frac{8gL}{9}}$$

Example: A small object is released at the top of a smooth fixed sphere due to which it slides on the sphere for some time and then breaks off (leaves) the sphere. Calculate the angle with upward vertical at the position where object leaves the sphere.

Soln. Let the angle with upward vertical is θ. Therefore height descended by object from its initial position.

$h=R(1-\cos\theta)$



from conservation of energy

loss in potential energy = gain in kinetic energy

$$mgR(1-\cos\theta)=\frac{1}{2}mv^2$$

$$\therefore v^2=2Rg(1-\cos\theta) \quad \ldots(a)$$

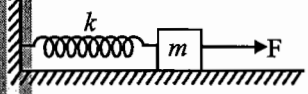
At the moment of leaving the sphere the object is still moving on circular path of radius R. Therefore

Net force toward centre $=\frac{mv^2}{R}$, or $mg\cos\theta=\frac{mv^2}{R}$

or $v^2=Rg\cos\theta \quad \ldots(b)$

from (a) and (b) we get $\cos\theta=\frac{2}{3}$ or $\theta=\cos^{-1}\frac{2}{3}$

Example: A block of mass m attached to a light spring of spring constant k lies in on a smooth horizontal plane as shown in figure. A horizontal force F is applied to the block.



Calculate maximum speed of the block.

Also calculate maximum elongation in spring.

Soln. When force is applied, the block accelerates for some time then it decelerates. Therefore acceleration of block is initially positive, then it momentarily becomes zero and finally becomes negative till the block stops. Speed of block is maximum at the instant when its acceleration is zero.

$a_x=0 \Rightarrow F_x=0$

$$\therefore F-kx=0 \Rightarrow x=\frac{F}{k}$$

Thus speed is maximum when elongation in spring is equal to $\frac{F}{k}$

To find maximum speed let us use work energy theorem.

Total work done = change in kinetic energy

$$W_{Spring}+W_{Force}=\frac{1}{2}mv^2-0 \text{ or } \int_0^{F/k} kx\,dx\cos 180^\circ+\int_0^{F/k} F\,dx=\frac{1}{2}mv^2$$

$$-\frac{F^2}{2k}+\frac{F^2}{k}=\frac{1}{2}mv^2 \quad \text{or} \quad v=\sqrt{\frac{F^2}{mk}}$$

Let x_0 be the maximum elongation

When elongation becomes maximum the block again comes to rest.

CAREER ENDEAVOUR

Therefore from work energy theorem we get,

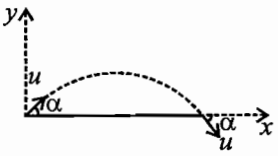
$$W_{Spring} + W_{Force} = 0 - 0, \quad \int_0^{x_0} kx dx \cos 180° + \int_0^{x_0} F dx = 0$$

$$-\frac{1}{2}kx_0^2 + Fx_0 = 0, \quad \therefore x_0 = \frac{2F}{k}$$

Impulse: It is equal to change in momentum of object $\vec{I} = \vec{p}_{Final} - \vec{p}_{initial}$ or $\vec{I} = \int \vec{F} dt$

Example: A projectile is thrown with initial velocity u at angle α with horizontal. Calculate total impulse on the projectile due to gravitational force:

Soln. Initial momentum of projectile $\vec{p}_{initial} = mu(\cos\alpha\hat{i} + \sin\alpha\hat{j})$



final momentum of projectile $\vec{p}_{final} = mu(\cos\alpha\hat{i} - \sin\alpha\hat{j})$

Impulse on the projectile, $\vec{I} = \vec{p}_{final} - \vec{p}_{initial} = -2mu\sin\alpha\hat{j}$

Conservation of momentum: Total momentum of a system remains conserved if there is no external force on the system. This conclusion comes directly from Newton's second law $\vec{F} = \frac{d\vec{p}}{dt}$, if $\vec{F} = 0$, $\frac{d\vec{p}}{dt} = 0$

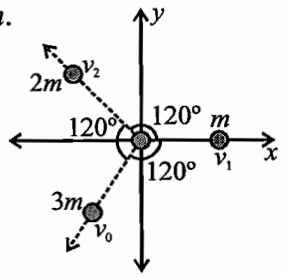
$\Rightarrow \vec{p}$ = constant

Note: If two particles are interacting with each other, momentum of each particle changes with time, however sum of momenta of the two particles remains constant. For example in rocket propulsion, momentum of rocket increases with time, however sum of momenta of fuel and rocket remains constant.

Example: An object initially kept at rest breaks up into three fragments of mass ratio 1 : 2 : 3. If speed of heaviest fragment just after explosion is v_0 what is speed of lightest fragment. It is given that the three fragments move in same plane and make equal angle with each other.

Soln. There is no external force on the object. Therefore its momentum remains constant. Initial momentum is zero. Therefore vector sum of momenta of three fragments must be zero.

$\vec{p} = 0 \Rightarrow p_x = 0, p_y = 0$. Let the masses of three fragments be m, $2m$ and $3m$.



Let v_1 and v_2 be speed of fragments of masses m and $2m$.

It is given that speed of $3m$ is v_0.

$p_x = 0$

$\Rightarrow \quad mv_1 - 2mv_2\cos 60 - 3mv_0\cos 60 = 0$

$$\therefore \quad v_1 - v_2 = \frac{3v_0}{2} \qquad \text{... (a)}$$

$p_y = 0 \qquad \Rightarrow \quad 2mv_2\sin 60 - 3mv_0\sin 60 = 0$

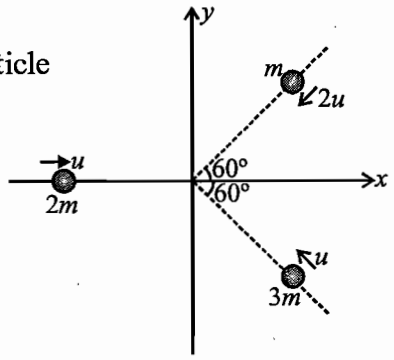
$$\therefore \quad v_2 = \frac{3v_0}{2} \qquad \text{... (b)}$$

from (a) $v_1 = \frac{3v_0}{2} + v_2 = \frac{3v_0}{2} + \frac{3v_0}{2} = 3v_0$

Therefore speed of the lightest fragment is $3v_0$.

Example: Three particles moving as shown in figure, collide at origin and coalesce to form a single particle. Calculate speed of the new particle formed.

Soln. There is no external force on the system of three particles. Therefore momentum must be conserved or each component of moment must be conserved.



Let V_x and V_y be x and y components of velocity of new particle

Therefore, $p_{x_{initial}} = p_{x_{final}}$

$$2mu - m2u\cos 60 - 3mu\cos 60 = 6mV_x$$

$$\therefore V_x = -\frac{u}{12} \text{ and } p_{y_{initial}} = p_{y_{final}}$$

$$-m2u\sin 60 + 3mu\sin 60 = 6mV_y \quad \therefore V_y = \frac{\sqrt{3}u}{12}$$

Therefore, velocity of new particle $= \sqrt{V_x^2 + V_y^2} = \frac{u}{12}\sqrt{3+1} = \frac{u}{6}$

CAREER ENDEAVOUR

Stability Analysis

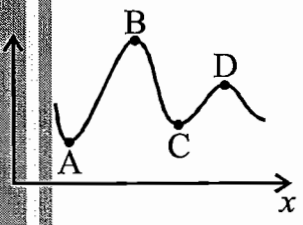
By stability analysis we mean finding equilibrium positions and investigating whether the given equilibrium is stable or unstable. It is easier to do stability analysis through potential rather than force. Therefore we will try to write potential of given system to discuss equilibrium whenever required.

Equilibrium criteria in one dimension:

If $V(x)$ be the potential under which a particle is moving then force acting on the particle is $F_x = -\frac{dV(x)}{dx}$

at equilibrium point $F_x = 0$. Therefore, condition for equilibrium in terms of potential becomes, $\frac{dV(x)}{dx} = 0$

In $V(x)$ versus x graph $\frac{dV}{dx} = 0$ at the points where tangent to the curve is parallel to x axis. Therefore in the figure shown below point A, B, C and D are equilibrium points.



Stable equilibrium point: A point is stable equilibrium point if a particle at this point when displaced towards right experiences force towards left and vice versa. That is the force tries to bring it back.

Therefore at stable equilibrium point $\frac{dF_x}{dx} < 0$

$\therefore \frac{d^2V(x)}{dx^2} > 0$ (condition for minimum)

Thus, a stable equilibrium point is a minimum on $V(x)$ versus x plot.
Therefore points A and C in the figure shown above are stable point.

Unstable equilibrium point: In this case a particle at equilibrium point is when displaced towards right it experiences force also in rightward direction. That is the force tries to displace the particle away from equilibrium point. Therefore at unstable equilibrium point, $\frac{dF_x}{dx} > 0$

$\therefore \frac{d^2V(x)}{dx^2} < 0$ (condition of maximum)

Thus, an unstable point is a maximum on $V(x)$ versus x plot. Therefore points B and D in the figure shown above are unstable points.

Frequency of oscillation: We know a particle of mass m moving under potential $V(x) = V_0 + \frac{1}{2}kx^2$ has frequency of oscillation $\omega = \sqrt{\frac{k}{m}}$, about stable equilibrium point. For a particle moving under some arbitrary potential $V(x)$, if x_0 be the stable equilibrium point, then we can expand $V(x)$ about x_0 using Taylor's series expansion as,

$$V(x) = V(x_0) + (x - x_0)\left.\frac{dV}{dx}\right|_{x=x_0} + \frac{(x-x_0)^2}{2}\left.\frac{d^2V}{dx^2}\right|_{x=x_0} + \frac{(x-x_0)^3}{\lfloor 3}\left.\frac{d^3V}{dx^3}\right|_{x=x_0} + \ldots\ldots$$

where $(x-x_0)$ is displacement from stable equilibrium point.

At this point, $\left.\frac{dV}{dx}\right|_{x=x_0}=0$, therefore, if $(x-x_0)$ be small then

$$V(x)=V(x_0)+\frac{(x-x_0)^2}{2}\left.\frac{d^2V}{dx^2}\right|_{x=x_0} \quad \text{(higher power terms neglected)}$$

or $V(x)=V(x_0)+\frac{1}{2}k(x-x_0)^2$

Therefore frequency of oscillation about stable equilibrium is

$\omega=\sqrt{\frac{k}{m}}$ where, $k=\left.\frac{d^2V}{dx^2}\right|_{x=x_0}$, k is called force constant.

Equilibrium criteria in two dimension: If a particle is moving under a two dimensional potential $V(x,y)$ then, at equilibrium point,

$$\left.\frac{\partial}{\partial x}V(x,y)\right|_{x_0,y_0}=0,\ \left.\frac{\partial}{\partial y}V(x,y)\right|_{x_0,y_0}=0$$

for stable equilibrium point (minimum)

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0}>0,\ \left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0}>0$$

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0}\cdot\left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0}>\left[\left.\frac{\partial^2}{\partial x\partial y}V(x,y)\right|_{x_0,y_0}\right]^2$$

For unstable equilibrium point (maximum)

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0}<0,\ \left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0}<0$$

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0}\cdot\left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0}>\left[\left.\frac{\partial^2}{\partial x\partial y}V(x,y)\right|_{x_0,y_0}\right]^2$$

for saddle point:

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0}\cdot\left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0}<\left[\left.\frac{\partial^2}{\partial x\partial y}V(x,y)\right|_{x_0,y_0}\right]^2$$

Example: A particle of mass m is moving under a one dimensional potential $V(x)=-ax+bx^2$ where $a>0$ and $b>0$. Find the equilibrium points and find frequency of oscillation about the stable equilibrium.

Soln. At equilibrium point, $\left.\frac{dV}{dx}\right|_{x=x_0}=0$

$$\therefore\ -a+2bx_0=0 \quad \text{or} \quad x_0=\frac{a}{2b}$$

Thus, $x_0 = \frac{a}{2b}$ is an equilibrium point.

To know whether it is stable or unstable equilibrium point let us calculate second derivative of potential at equilibrium point.

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 2b > 0$$

Therefore $x_0 = \frac{a}{2b}$ is stable equilibrium point, force constant $k = \left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 2b$

Therefore, frequency of oscillation is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2b}{m}}$$

Example: Potential corresponding to force between the atoms of a diatomic molecule is $V(r) = \frac{a}{r^{12}} - \frac{b}{r^6}$ where a and b are positive constants and r is separation between the atoms. Calculate bond length for stable configuration and also calculate frequency of oscillation of atoms if mass of each atom be m.

Soln. For stable configuration $\left.\frac{dV}{dr}\right|_{r=r_0} = 0$

$$-\frac{12a}{r_0^{13}} + \frac{6b}{r_0^7} = 0 \Rightarrow r_0 = \left(\frac{2a}{b}\right)^{1/6}$$

Therefore bond length for stable configuration is $\left(\frac{2a}{b}\right)^{1/6}$

Force constant $k = \left.\frac{d^2V}{dr^2}\right|_{r=r_0} = \frac{12\times13a}{r_0^{14}} - \frac{6\times7b}{r_0^8} = \frac{1}{r_0^8}\left(\frac{12\times13a}{r_0^6} - 6\times7b\right)$

$$= \left(\frac{b}{2a}\right)^{8/6}\left(\frac{12\times13a}{2a/b} - 42b\right) = \left(\frac{b}{2a}\right)^{4/3}.36b = \frac{18}{2^{1/3}}\cdot\frac{b^{7/3}}{a^{4/3}}$$

Reduced mass of system $\mu = \frac{m.m}{m+m} = m/2$

Frequency of oscillation $\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{18}{2^{1/3}}\cdot\frac{b^{7/3}}{m/2\,a^{4/3}}} = 6\left(\frac{b^7}{2m^3\,a^4}\right)^{1/6}$

Example: A particle of mass 'm' is moving under potential $V(x) = ax^3 - bx^2$. Initially the particle is at rest at stable point. What minimum speed be given to the particle so that it reaches unstable point. Plot the potential versus x.

Soln. For equilibrium point $\left.\frac{dV}{dx}\right|_{x=x_0} = 0$

$3ax_0^2 - 2bx_0 = 0 \Rightarrow x_0 = 0, \frac{2b}{3a}$

$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 6ax_0 - 2b, \quad \left.\frac{d^2V}{dx^2}\right|_{x=0} = -2b < 0, \therefore x_0 = 0$ is unstable point

$\left.\frac{d^2V}{dx^2}\right|_{x=\frac{2b}{3a}} = 2b > 0, \therefore x_0 = \frac{2b}{3a}$ is stable point

To calculate speed let us apply conservation of energy.
Total energy initial = Total energy final

(Kinetic Energy + Potential Energy) at $x = \frac{2b}{a}$ = (Kinetic Energy + Potential Energy) at $x = 0$

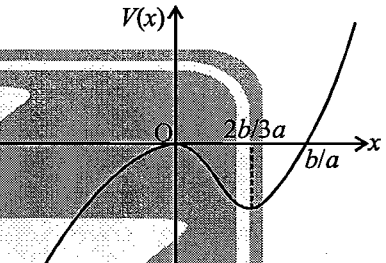
for minimum speed (u) at stable point the particle will just reach unstable point and stops there.

$\therefore \frac{1}{2}mu^2 + V\left(x = \frac{2b}{3a}\right) = \frac{1}{2}m.0^2 + V(x=0)$

$\frac{1}{2}mu^2 + a\left(\frac{2b}{3a}\right) - b\left(\frac{2b}{3a}\right)^2 = 0 + 0$

$u^2 = -\frac{2}{m}\left(\frac{2b}{3a}\right)^2\left(\frac{2b}{3} - b\right) = \frac{2}{m}\cdot\frac{4b^2}{9a^2}\cdot\frac{b}{3} = \frac{8b^2}{27ma^2}$

$\therefore u = \sqrt{\frac{8b^2}{27ma^2}}$



Example : A particle is moving under potential $V(r) = \frac{a}{r^2} - \frac{b}{r}$. Calculate the minimum value of potential energy.

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Soln. For potential to be minimum $\left.\frac{dV}{dr}\right|_{r=r_0} = 0$

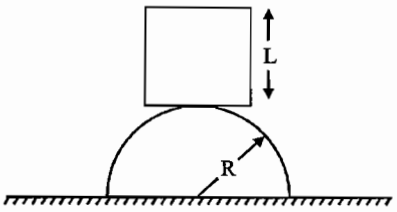
$-\frac{2a}{r_0^3} + \frac{b}{r_0^2} = 0 \quad \therefore r_0 = \frac{2a}{b}$

$\left.\frac{d^2V}{dr^2}\right|_{r=r_0} = \frac{6a}{r_0^4} - \frac{2b}{r_0^3} = \frac{1}{r_0^3}\left(\frac{6a}{r_0} - 2b\right) = \frac{1}{r_0^3}\left(\frac{6a}{2a/b} - 2b\right) = \frac{4b}{r_0^3} > 0$

Therefore at $r = r_0$ potential is minimum

$\therefore \quad V_{min} = V(r_0) = \frac{1}{r_0}\left(\frac{a}{r_0} - b\right) = -\frac{b^2}{4a}$

Example: A cube is placed on the top of a fixed hemisphere as shown in figure. What should be relation between length of side of cube and radius of hemisphere so that cube has stable equilibrium.



Soln. To discuss equilibrium of cube we first write its potential energy as function of angle from its equilibrium position. As shown in the figure below, initially point A was in contact with spherical surface but now in displaced position point B is in contact. Therefore,

$AB = R\theta$

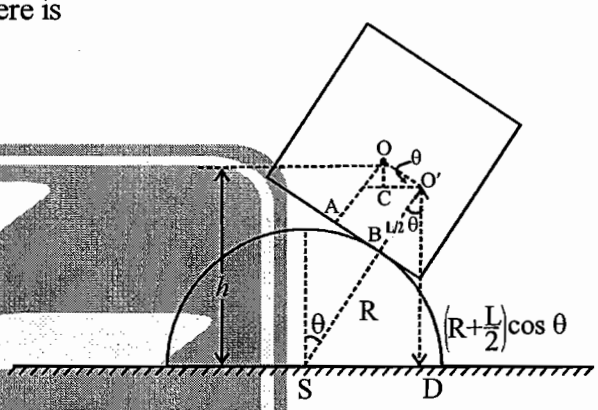
Height of centre of cube from centre level of hemisphere is

$h = OC + O'D$

$= OO'\sin\theta + O'S\cos\theta$

$= AB\sin\theta + (O'B + BS)\cos\theta$

$= R\theta\sin\theta + \left(\frac{L}{2} + R\right)\cos\theta$



Potential energy of the cube

$V(\theta) = Mgh \quad = Mg\left[R\theta\sin\theta + \left(\frac{L}{2} + R\right)\cos\theta\right]$

$\theta = 0$ is equilibrium position of the cube. For this position to be stable equilibrium position, $\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0} > 0$

$\therefore \frac{d^2}{d\theta^2} Mg\left[R\theta\sin\theta + \left(\frac{L}{2} + R\right)\cos\theta\right]_{\theta=0} > 0$

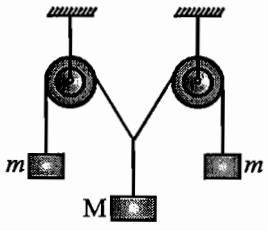
or $\frac{d}{d\theta}\left[R\theta\cos\theta + R\sin\theta - \left(\frac{L}{2} + R\right)\sin\theta\right]_{\theta=0} > 0$

or $\left[R\cos\theta - R\theta\sin\theta + R\cos\theta - \left(\frac{L}{2} + R\right)\cos\theta\right]_{\theta=0} > 0$

or $\left[2R - \left(\frac{L}{2} + R\right)\right] > 0 \quad \therefore R - \frac{L}{2} > 0$ or $2R > L$

Thus, cube can be in stable equilibrium position if its side length is less than diameter of hemisphere.

Example: For the mass pulley system shown in figure, what should be relation between m and M so that system remains in stable equilibrium position. Pulley are smooth and strings are tight and inextensible.



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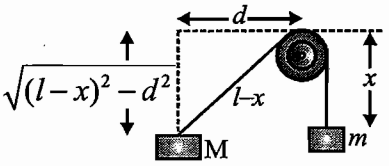
Soln. The pulleys are fixed. Therefore we can write potential energy of the system by specifying position of blocks with respect to pulleys.

Let 'l' be length of string and 'd' be the half distance between two pulleys. Therefore, l and d are constants. If x be distance of m below the pulley as shown in figure then potential energy of system is

$$V(x) = -2mgx - Mg\sqrt{(l-x)^2 - d^2} \qquad \therefore \frac{dV}{dx} = -2mg + \frac{Mg(l-x)}{\sqrt{(l-x)^2 - d^2}}$$

for equilibrium $\left.\frac{dV}{dx}\right|_{x=x_0} = 0$ $\qquad \therefore -2mg + \frac{Mg(l-x_0)}{\sqrt{(l-x_0)^2 - d^2}} = 0$

or $\frac{2m}{M} = \frac{l-x_0}{\sqrt{(l-x_0)^2 - d^2}}$ ... (a)



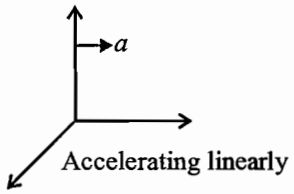
$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = \frac{Mgd^2}{\left[(l-x_0)^2 - d^2\right]^{3/2}} > 0 \text{ for all values of } d > 0$$

since, $\frac{l-x_0}{\sqrt{(l-x_0)^2 - d^2}} > 1$ $\therefore$ from (a) $\frac{2m}{M} > 1$ or $2m > M$

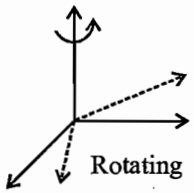
CAREER ENDEAVOUR

Non-inertial frames and pseudoforces

A non-inertial frame is that which is accelerated. It might be accelerating translationally or might be rotating. In the figure shown below two non-inertial fames are shown.



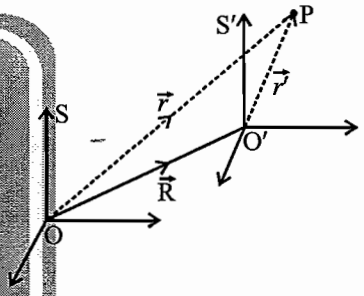
Accelerating linearly



Rotating

When observation is made from non-inertial frame then it appears that Newton's laws of motion are not being obeyed. The reason is that it appears as if in addition to the known forces some unknown forces are also acting the system. The unknown forces are called pseudoforces or fictitious forces.

Linearly accelerated frame: Let us consider a frame S' which is linearly accelerated with acceleration $\vec{a}_0$ with respect to an inertial frame S.



From figure, $\vec{R}$ = position vector of S' w.r.t. S

$$\therefore \ \vec{a}_0 = \frac{d^2\vec{R}}{dt^2}$$

If $\vec{F}$ be the known forces acting on a particle as measured in S frame

then, $\vec{F} = \frac{md^2\vec{r}}{dt^2} = m\vec{a}$, where $\vec{a}$ is acceleration of particle in S frame.

From figure, $\vec{r}' = \vec{r} - \vec{R}$ or $\frac{d^2\vec{r}'}{dt^2} = \frac{d^2\vec{r}}{dt^2} - \frac{d^2\vec{R}}{dt^2}$

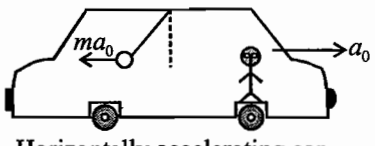
If $\vec{a}'$ be acceleration of particle in S' then from above relation we get $\vec{a}' = \vec{a} - \vec{a}_0$

Force on the particle as seen from S'

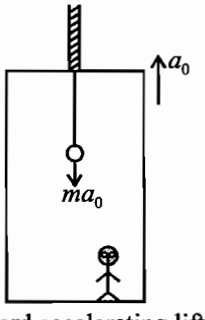
$\vec{F}' = m\vec{a}' = m(\vec{a} - \vec{a}_0)$ $\quad \therefore \vec{F}' = \vec{F} - m\vec{a}_0$

from above relation we see that net force in S' frame consists of a term $-m\vec{a}_0$ in addition to the known forces $\vec{F}$, clearly $-m\vec{a}_0$ is coming into picture because of acceleration of $\vec{S}'$ frame. Therefore in linearly accelerated frame, pseudo force = $-m\vec{a}_0$

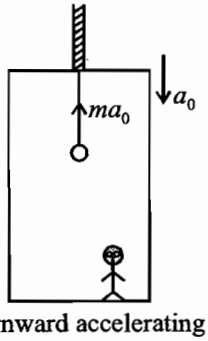
minus sign shows that pseudo force acts opposite to the direction of acceleration of the frame.
Pseudo force in some linearly accelerated frames is shown in figure.



Horizontally accelerating car



Upward accelerating lift



Downward accelerating lift

CAREER ENDEAVOUR

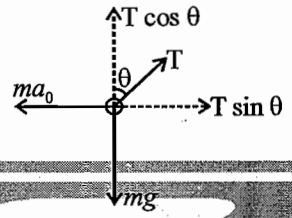
Note: The word pseudo force is used by an observer standing on the non-inertial frame. An observer in inertial frame cannot use the word pseudo force. However observer on non-inertial frame finds that pseudo force acts on all objects including the observer itself irrespective of location of the objects. That is objects may or may not be on non-inertial frame.

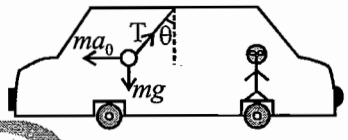
Example: A simple pendulum consisting of a point mass m and a light string of length 'l' is suspended from the roof of a car which is moving horizontally with constant acceleration a_0. Find the angle made by string with vertical.

Soln. As seen from car the pseudo force ma_0 will act on the bob opposite to the direction of acceleration of car due to which string will deflect from its vertically downward position.

Let θ be the angle made by string with downward vertical in the equilibrium position. The known forces acting on the bob are mg and tension T as shown in figure.

For equilibrium of bob, we must have,



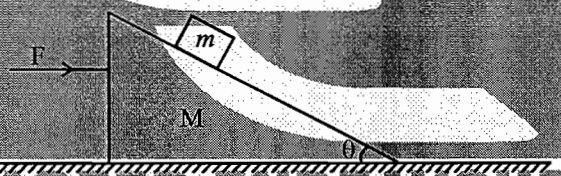


$$T\sin\theta = ma_0, \quad T\cos\theta = mg$$

Dividing the above two relations we get.

$$\tan\theta = \frac{a_0}{g} \text{ or } \theta = \tan^{-1}\frac{a_0}{g}$$

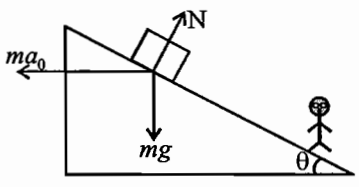
Example: A small block of mass 'm' lies on a wedge of mass M as shown in figure. All the contact surfaces are smooth. When a horizontal force F is applied to the wedge, the block does not slide on the wedge. What must be value of F.

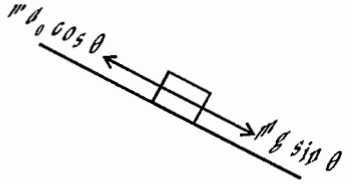


CAREER ENDEAVOUR

Soln. Horizontal acceleration of the system is $a_0 = \dfrac{F}{M+m}$

According to question, the block does not slide on the wedge, therefore if block is seen from the reference frame of the wedge it will appear stationary. The wedge has a linear acceleration a_0 therefore if observation is made from the wedge frame a pseudo force $-ma_0$ must be applied on the block as shown in figure. Block is stationary on the wedge so component of forces parallel to the inclined plane must cancel.

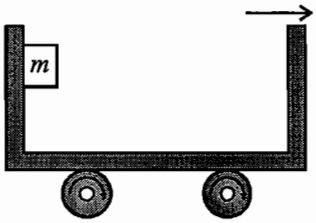




Therefore, $mg\sin\theta = ma_0\cos\theta$ or $\tan\theta = \dfrac{a_0}{g} = \dfrac{F}{(M+m)g}$

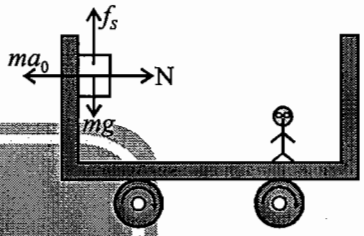
or $F = (M+m)g\tan\theta$

Example: A trolley is moving horizontally with a constant acceleration as shown in figure. A block remains stationary on the vertical wall of the trolley. If coefficients of static friction between the wall and block is μ what must be acceleration of the trolley.



Soln. As seen from the trolley's frame the different forces acting on the block are shown in figure. Let a_0 be acceleration of trolley. f_S and N are static friction and normal reaction respectively. For equilibrium of block we must have

$f_S = mg$ and $N = ma_0$



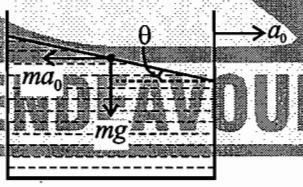
We know that $f_S \le \mu N \quad \therefore mg \le \mu ma_0 \quad \therefore a_0 \ge \frac{g}{\mu}$

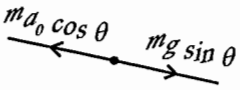
Therefore acceleration of trolley must either be equal to or greater than g/μ.

Example: A vessel containing water is linearly accelerated due to which surface of water does not remain horizontal. If a_0 be acceleration of vessel what is angle made by water surface with horizontal.

Soln. Due to pseudo force water will move opposite to direction of acceleration. Therefore surface of water becomes inclined as shown in figure. Consider a particle of water of mass m right at the surface. If θ be the angle made by water surface with horizontal then component of pseudo force $-ma_0$ and mg parallel to the surface are as shown in figure.

For equilibrium these forces must cancel.



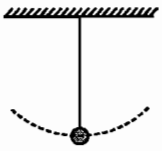


Therefore $ma_0 \cos\theta = mg \sin\theta$ or $\tan\theta = \frac{a_0}{g}$

$\theta = \tan^{-1}\frac{a_0}{g}$

Time period of simple pendulum in linearly accelerated frame: We know that time period of simple pendulum in an inertial frame when point of suspension is stationary is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$



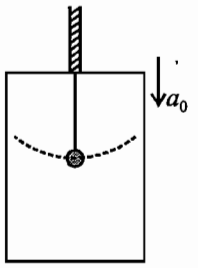
when simple pendulum is suspended in such a way that point of suspension moves with acceleration $\vec{a}_0$ then time period for small oscillation about the equilibrium point is given as

$$T = 2\pi\sqrt{\frac{l}{g_{eff}}}$$

where g_{eff} is effective value of acceleration due to gravity and it is given as $g_{eff} = |\vec{g} - \vec{a}_0|$

Special cases:

(i) Lift accelerating downward.

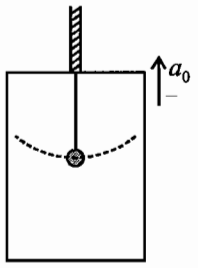


In this case $\vec{a}_0$ is parallel to $\vec{g}$.

Therefore $\therefore g_{eff} = g - a_0,\ a_0 < g$

$$\therefore T = 2\pi\sqrt{\frac{l}{g-a_0}}$$

(ii) Lift accelerating upward



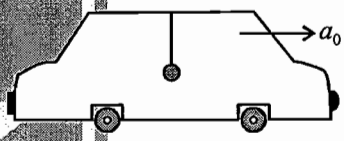
In this case $\vec{a}_0$ is opposite to $\vec{g}$

$$\therefore g_{eff} = g + a_0$$

$$\therefore T = 2\pi\sqrt{\frac{l}{g+a_0}}$$

(iii) Car accelerating horizontally.

In this case $\vec{a}_0$ is perpendicular to $\vec{g}$. Therefore



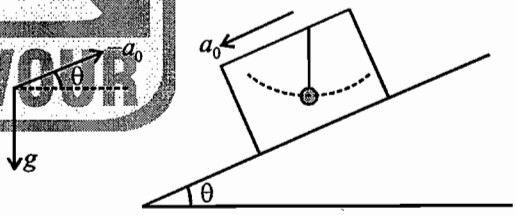
$$g_{eff} = \sqrt{g^2 + a_0^2}$$

$$\therefore T = 2\pi\sqrt{\frac{l}{\sqrt{g^2+a_0^2}}}$$

Example: A large box is placed on a smooth inclined plane of inclination θ due to which the box slides down. Simple pendulum is suspended from the top of the box inside it. Calculate time period of small oscillation of the pendulum.

Soln. Acceleration of box down the inclined plane is $a_0 = g\sin\theta$

Angle between $-a_0$ and g in this case is $90+\theta$



$$g_{eff} = |\vec{g} - \vec{a}_0|$$

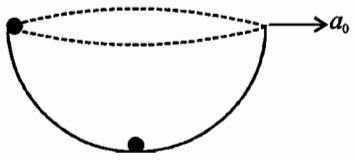
$$= \sqrt{g^2 + a_0^2 + 2ga_0\cos(90+\theta)}$$

$$= \sqrt{g^2 + g^2\sin^2\theta - 2g^2\sin^2\theta}$$

$$= g\cos\theta$$

$$\therefore T = 2\pi\sqrt{\frac{l}{g_{eff}}} = 2\pi\sqrt{\frac{l}{g\cos\theta}}$$

Example: A small particle of mass m is released at the edge of a smooth bowl as shown in figure. The bowl is accelerated horizontally with constant acceleration a_0. The particle slides along the surface of bowl. Calculate normal reaction on the particle when it reaches the bottom.

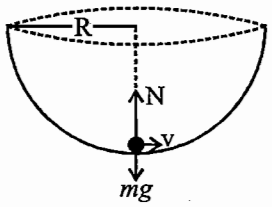


CAREER ENDEAVOUR

Soln. This problem gets simplified if we observe motion of the particle with respect to the bowl. In bowl's frame particle moves simply in vertical circle of radius equal to radius of bowl. And it is acted upon by mg, $-ma_0$ and normal reaction.

Let v be the speed of particle w.r.t. bowl at the lowest point. Therefore

$$N - mg = \frac{mv^2}{R}$$



or $N = mg + \frac{mv^2}{R}$ ... (a)

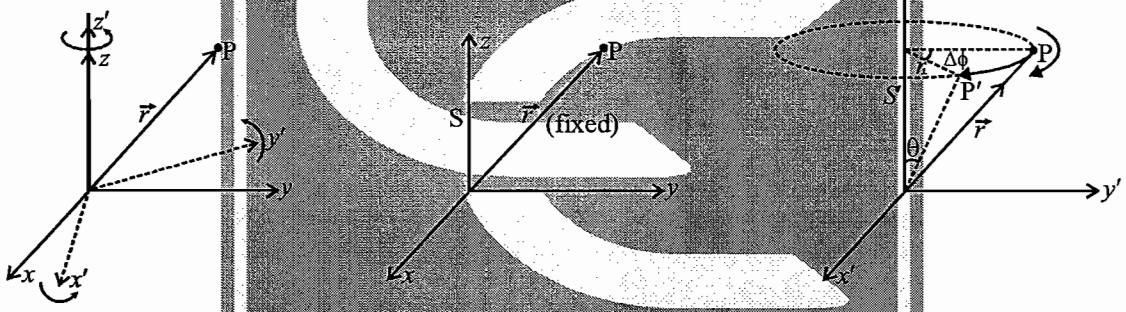
To calculate v let us apply work energy theorem.

Total work done by all forces = Change in kinetic energy

$$mgR - ma_0R = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{2(g-a_0)R}$$

from (a) $N = mg + \frac{m2(g-a_0)R}{R}$, $N = m(3g - a_0)$

Rotating Frame: Let S' frame is rotating with respect to S frame. Therefore a vector which is fixed in S frame will appear to be rotating in S' in a direction opposite to the rotation of S'.



For convenience we have assumed that origin and z-axes of S and S' coincide. In figures above we have shown the two frames together and a fixed vector as seen from the two frames separately.

As seen from S' a vector which is fixed in S frame describes a cone in S' frame. Therefore change in vector $\vec{r}$ in small time Δt as seen frame S' is

$= p\vec{p}'$

$= r_\perp \Delta\phi \hat{e} = r\sin\theta\omega\Delta t\hat{e}$ $\{\hat{e}$ is unit vector parallel to $p\vec{p}'\}$

$= (r\omega\sin\theta\hat{e})\Delta t = -\vec{\omega}\times\vec{r}\Delta t$

Thus a fixed vector $\vec{r}$ appears to change by an amount $-\vec{\omega}\times\vec{r}\Delta t$ in time Δt when looked at from a rotating frame. However if the vector $\vec{r}$ is not fixed in S frame and if it changes by $\Delta\vec{r}$ in S frame in time Δt then change in vector as seen frame S' will include extra change due to rotation also. Therefore in S' frame change in vector will be $\Delta\vec{r}' = \Delta\vec{r} - \vec{\omega}\times\vec{r}\Delta t$

$\therefore (\Delta\vec{r})_{\text{in } S'} = (\Delta\vec{r})_{\text{in } S} - \vec{\omega}\times\vec{r}\Delta t$

or $\left(\frac{\Delta\vec{r}}{\Delta t}\right)_{\text{in } S'} = \left(\frac{\Delta\vec{r}}{\Delta t}\right)_{\text{in } S} - \vec{\omega}\times\vec{r}$

In the limit $\Delta t \to 0$

$$\left(\frac{d\vec{r}}{dt}\right)_{\text{in } S'} = \left(\frac{d\vec{r}}{dt}\right)_{\text{in } S} - \vec{\omega} \times \vec{r}$$

The same concept can be applied to all vectors because all vectors will change in similar way when observed from rotating frame. For example when applied to angular momentum we get

$$\left(\frac{d\vec{L}}{dt}\right)_{\text{in } S'} = \left(\frac{d\vec{L}}{dt}\right)_{\text{in } S} - \vec{\omega} \times \vec{L}$$

Pseudo forces in rotating frame: Let us consider dynamics of a particle of mass m in S and S' frames. Let $\vec{r}(t)$ be instantaneous position vector of particle. Therefore in S frame (inertial frame)

Velocity of particle $\vec{v} = \left(\frac{d\vec{r}}{dt}\right)_{\text{in}S}$

Acceleration of particle $\vec{a} = \left(\frac{d\vec{v}}{dt}\right)_{\text{in}S}$

Net force on the particle $\vec{F} = m\left(\frac{d\vec{v}}{dt}\right)_{\text{in}S}$ where $\vec{F}$ is resultant of all known forces.

In S' frame (rotating frame)

Net force on the particle is $\vec{F}' = m\left(\frac{d\vec{v}'}{dt}\right)_{\text{in}S'}$

From relation $\left(\frac{d\vec{r}}{dt}\right)_{\text{in}S'} = \left(\frac{d\vec{r}}{dt}\right)_{\text{in}S} - \vec{\omega} \times \vec{r}$

We get $\vec{v}' = \vec{v} - \vec{\omega} \times \vec{r}$ and $\left(\frac{d\vec{v}'}{dt}\right)_{\text{in}S'} = \left(\frac{d\vec{v}'}{dt}\right)_{\text{in}S} - \vec{\omega} \times \vec{v}'$

$$\therefore \vec{F}' = m\left[\left(\frac{d\vec{v}'}{dt}\right)_{\text{in}S} - \vec{\omega} \times \vec{v}'\right] = m\left[\frac{d}{dt}(\vec{v} - \vec{\omega} \times \vec{r})_{\text{in}S} - \vec{\omega} \times \vec{v}'\right]$$

$$= m\left(\frac{d\vec{v}}{dt}\right)_{\text{in}S} - m\left(\frac{d\vec{\omega}}{dt} \times \vec{r}\right)_{\text{in}S} - m\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{\text{in}S} - m\vec{\omega} \times \vec{v}'$$

$$\vec{F}' = \vec{F} - m\frac{d\vec{\omega}}{dt} \times \vec{r} - m\vec{\omega} \times \vec{v} - m\vec{\omega} \times \vec{v}'$$

Since observation is being made with respect to the rotating frame therefore we would like to express $\vec{F}'$ in terms of velocity w.r.t. rotating frame i.e. $\vec{v}'$, therefore let us remove $\vec{v}$ by using $\vec{v} = \vec{v}' + \vec{\omega} \times \vec{r}$ Therefore

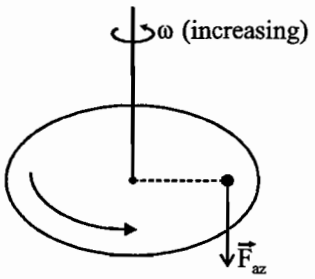
$$\vec{F}' = \vec{F} - m\frac{d\vec{\omega}}{dt} \times \vec{r} - m\vec{\omega} \times (\vec{v}' + \vec{\omega} \times \vec{r}) - m\vec{\omega} \times \vec{v}'$$

$$\boxed{\vec{F}' = \vec{F} - m\frac{d\vec{\omega}}{dt} \times \vec{r} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}'}$$

Thus, as seen from S', in addition to the a known force $\vec{F}$ some unknown forces also appear to be acting on the particle. These forces are called pseudo forces. In the following we discuss each term of pseudo forces one by one.

CAREER ENDEAVOUR

Azimuthal Force: It appears to act on the particles which are being observed from a rotating frame which has non-uniform angular velocity i.e. $\frac{d\omega}{dt} \neq 0$.



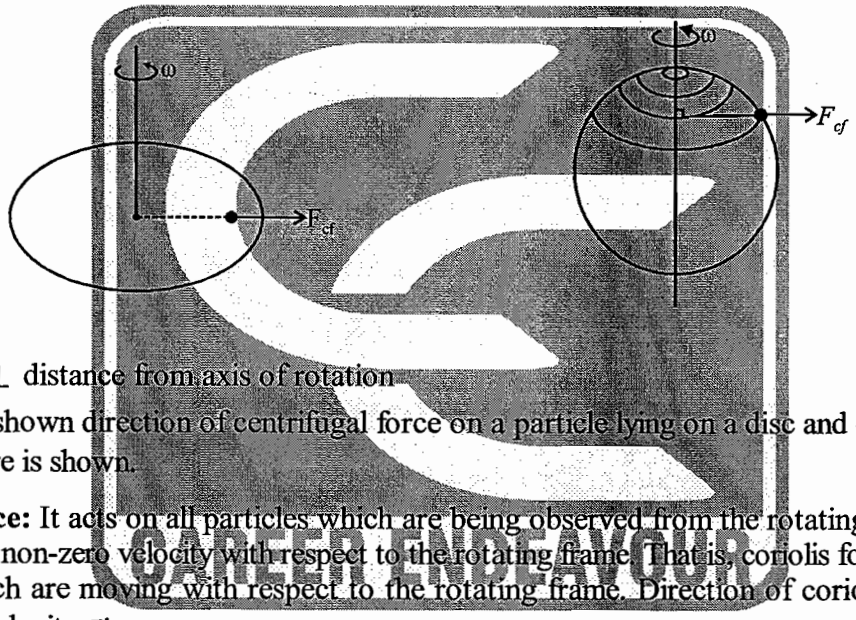
Its direction is tangential to the rotation of frame.

Azimuthal Force $= -m\frac{d\vec{\omega}}{dt} \times \vec{r} = \vec{F}_{az}$

In the figure direction of azimuthal force on a particle lying on a non-uniformly rotating disc is shown.

Centrifugal Force: It acts on all particles which are being observed from the rotating frame, irrespective of state of the particle. Direction of centrifugal force is perpendicular to the axes of rotation and away from it.

Centrifugal Force $= -m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{F}_{cf}$ or $|\vec{F}_{cf}| = m\omega^2 r_{\perp}$,



where $r_{\perp}$ is $\perp$ distance from axis of rotation.

In the figure shown direction of centrifugal force on a particle lying on a disc and one lying surface of a rotating sphere is shown.

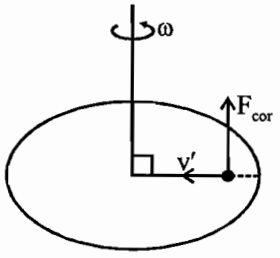
Coriolis Force: It acts on all particles which are being observed from the rotating frame provided that particles have non-zero velocity with respect to the rotating frame. That is, coriolis force acts only on those particles which are moving with respect to the rotating frame. Direction of coriolis force depends on direction of velocity $\vec{v}'$.

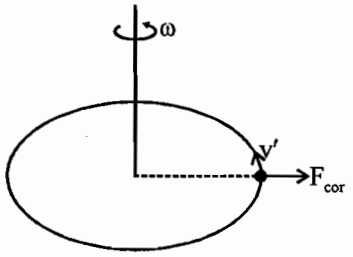
Coriolis force $= -2m\vec{\omega} \times \vec{v}' = \vec{F}_{cor}$

or $\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega}$

$\vec{F}_{cor}$ is $\perp$ to both $\vec{\omega}$ and $\vec{v}'$ and its direction can be found by using right hand thumb rule of cross product of two vectors.

Direction of coriolis force on an insect which is moving along the radius of a rotating disc and along rim is shown in figure.





Earth: A non-inertial (rotating) frame: Earth spins about its axis, therefore it is a non-inertial frame.

Angular velocity of earth's spinning motion is $\omega = \frac{2\pi}{T} = \frac{2\pi}{24h} = 7.27 \times 10^{-5}$ rad/sec

Due to rotations of earth following effects arise.

(a) All objects on earth's surface experience centrifugal force which is directed away from the axis of rotation.

(b) All objects moving on earth's surface experience coriolis force in addition to centrifugal force.

(c) An observer standing on earth's surface notices that on all objects a centrifugal and a coriolis force (if $v' \neq 0$) act irrespective of the location of the object. That is the object which is being observed by the observer may or may not be on the earth's surface. For example we on the earth, see that the sun rises and sets. If we analyse the dynamics of the sun we will find that sun experiences a centrifugal force away from the earth and a coriolis force towards the earth.

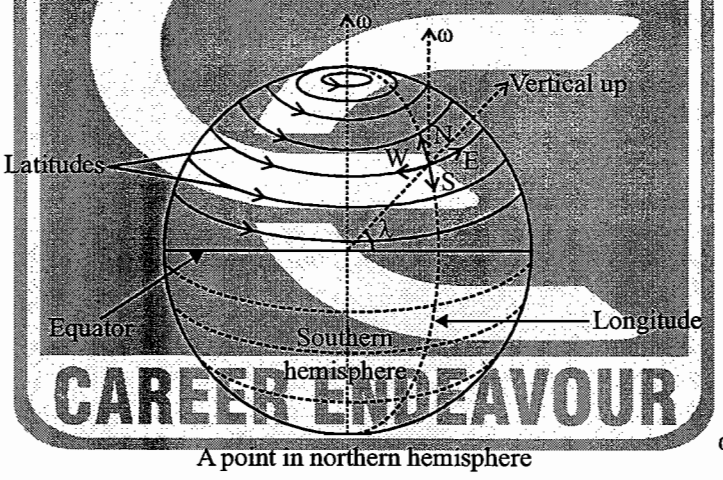
Coordinate system on earth's surface: To analyse dynamics of objects from earth's surface we define local cartesian coordinate system as follows:

+X-axis : Along local east, –X axis: Along local west

+Y-axis : Along local north, –Y axis: Along local south

+Z-axis : Along local vertical upward, –Z axis: Along local vertical downward

Therefore angular velocity vector of earth's spinning motion lies in Y-Z plane as shown in figure.



A point in northern hemisphere

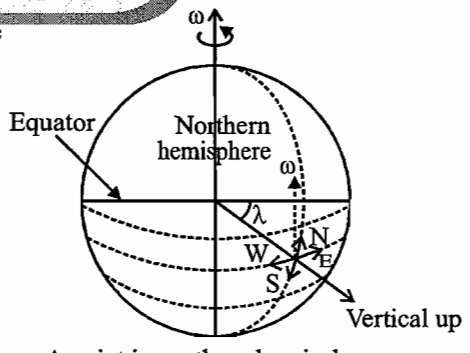
Therefore, at any point in northern hemisphere,

$\vec{\omega} = \omega(\cos\lambda\hat{j} + \sin\lambda\hat{k})$

where λ is latitude of the point.

And at any point in southern hemisphere

$\vec{\omega} = \omega(\cos\lambda\hat{j} - \sin\lambda\hat{k})$



A point in southern hemisphere

Coriolis force on a freely falling object: An object falls vertically downward therefore velocity of object is in negative z-direction,

$\therefore \vec{v}' = -v'\hat{k}$

In northern hemisphere: $\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega} = -2mv'\omega\hat{k} \times (\cos\lambda\hat{j} + \sin\lambda\hat{k}) = +2mv'\omega\cos\lambda\hat{i}$

Coriolic force is directed along +X direction, i.e. towards east. Therefore a freely falling object deviates towards east in northern hemisphere.

In southern hemisphere: $\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega} = -2mv'\omega\hat{k} \times (\cos\lambda\hat{j} - \sin\lambda\hat{k}) = 2mv'\omega\cos\lambda\hat{i}$

In southern hemisphere also, a freely falling object deviates towards east.

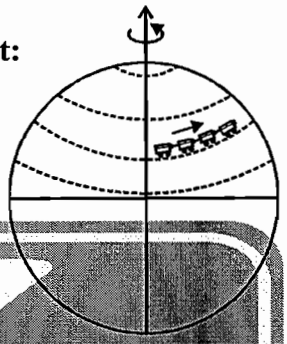
Coriolis force on river water: Consider a river flowing from north to south direction in northern hemisphere.

Therefore, velocity of water, $\vec{v}' = -v'\hat{j}$

Coriolis force on water, $\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega} = -2mv'\omega\hat{j} \times (\cos\lambda\hat{j} + \sin\lambda\hat{k}) = -2mv'\omega\sin\lambda\hat{i}$

Coriolis force on water is directed towards west. Due to which west bank of river flowing from north to south in northern hemisphere undergoes more erosion than east bank.

Coriolis force on a train moving from west to east:



Velocity of train, $\vec{v}' = v'\hat{i}$

Fcoriolis force on train,

$\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega}$

$= 2mv'\omega\hat{i} \times (\cos\lambda\hat{j} + \sin\lambda\hat{k})$

$= 2mv'\omega(\cos\lambda\hat{k} - \sin\lambda\hat{j})$

Therefore lateral force on the track of train due to coriolis force is $2mv\omega\sin\lambda$.

Coriolis and centrifugal force on the sun as seen from the earth: The sun appears to be rising and setting because of rotation of earth about its own axis. A person on the earth finds the sun to be moving. This is exactly the same effect as seen by a person sitting in a moving car who sees everything moving past the car.

Let us assume that axis of rotation of earth shown here is perpendicular to the plane of paper. Therefore as seen from the earth the sun appears to be rotating as shown in figure. And velocity of the sun with respect to the earth is

$v' = \omega r$, r = distance of sun from centre of earth

$\therefore$ Centrifugal force is always away from the axis of rotation. The direction of coriolis force can be found using right hand thumb rule and it is found to be towards the centre of earth.

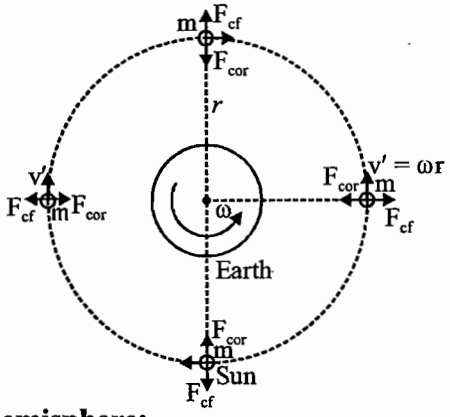
CAREER ENDEAVOUR

$F_{cor} = |2m\vec{v}' \times \vec{\omega}| = 2m\vec{v}'\omega\sin 90° = 2mv'\omega$

$= 2m\omega^2 r$

$F_{cf} = m\omega^2 r_\perp$, $r_\perp = r$

$\therefore F_{cf} = m\omega^2 r$



Eastward deviation of a freely falling object in northern hemisphere:

Suppose a particle falls from a height h at a place where latitude is λ.

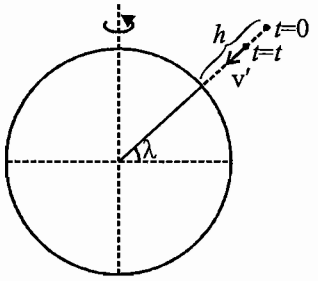
Here we assume that $h \ll R_e$

Velocity of particle at time t is $\vec{v}' = -gt\hat{k}$

Coriolis force on the particle

$$\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega} = -2m\omega g t\hat{k} \times (\cos\lambda\hat{j} + \sin\lambda\hat{k})$$

$$= 2m\omega g \cos\lambda t\hat{i}$$



Force is towards east so particle will deviate towards east, equation of motion of the particle in eastward direction is

$$F_{cor} = \frac{md^2x}{dt^2} \text{ or } \frac{d^2x}{dt^2} = 2\omega g \cos\lambda t$$

We have following initial condition which we use to solve above equation,

at $t = 0, x = 0, y = 0, z = h$ ... (a)

at $t = 0, \frac{dx}{dt} = 0, \frac{dy}{dt} = 0, \frac{dz}{dt} = 0$ ... (b)

integrating the above differential equation we get, $\frac{dx}{dt} = \omega g \cos\lambda t^2 + c_1$

using (b) we get $c_1 = 0$. Therefore $\frac{dx}{dt} = \omega g \cos\lambda t^2$

Integrating again we get, $x = \frac{1}{3}\omega g \cos\lambda t^3 + c_2$, using (a) we get, $c_2 = 0$

$\therefore$ $\boxed{x = \frac{1}{3}\omega g \cos\lambda t^3}$

where t is total time taken to reach earth surface

$\therefore$ $t = \sqrt{\frac{2h}{g}}$

$\therefore$ $\boxed{x = \frac{1}{3}\omega g \cos\lambda \left(\frac{2h}{g}\right)^{3/2}}$

CAREER ENDEAVOUR

If a particle falls from a height of 100 metre at equator ($\lambda = 0$) then using above equation we get.

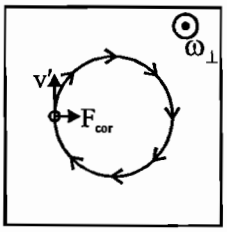
$$x = \frac{1}{3} \times 7.27 \times 10^{-5} \times 9.8 \times \left(\frac{2 \times 100}{9.8}\right)^{3/2} \simeq 2.19 \approx 2.2\text{cm}$$

Note: The expression derived for eastward deviation is an approximate one. In the derivation we have assumed that particle is uniformly accelerated in downward direction which is not true, when particle attains a velocity in X(each direction) there arises a component of coriolis force in vertical direction, due to which downward acceleration becomes a function of v_x'. Because of v_x' component of velocity, particle also experiences a force in southward direction. Therefore a freely falling object in northern hemisphere actually deviates in south-east direction.

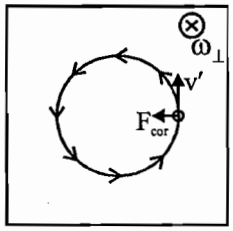
Circular motion on earth's surface due to coriolis force: Direction of coriolis force is perpendicular to the velocity of particle, therefore in absence of any other force like gravity, friction etc. a particle will move in circular path in the same manner as a charge particle does due to magnetic force.

Let us suppose that there is no friction and a particle is thrown horizontally on the surface of earth with speed v'. Let λ be the latitude of place. Therefore, $\vec{\omega} = \omega(\cos\lambda\hat{j} \pm \sin\lambda\hat{k})$ we use + or – sign depending on the case whether we are in northern or southern hemisphere. Component of $\vec{\omega}$ perpendicular the plane is $\pm\omega\sin\lambda$. Therefore a particle thrown horizontally will deviate in opposite direction in the two hemispheres,

as shown below:



Horizontal plane in N-hemisphere



Horizontal plane in S-hemisphere

Thus, circular path is clock wise in N-hemisphere and anticlockwise in S-hemisphere.
If r be the radius of circle then,

horizontal component of $F_{cor} = \frac{mv'^2}{r}$ or $2mv'\omega_{\perp}\sin 90° = \frac{mv'^2}{r}$

or $2mv'\omega\sin\lambda = \frac{mv'^2}{r}$ $\therefore$ $\boxed{r = \frac{v'}{2\omega\sin\lambda}}$

If the velocity with which particle is thrown is very small say 1 mm/sec then radius of circular path near pole $(\lambda = 90°)$ will be

$$r = \frac{1\times10^{-3}}{2\times7.27\times10^{-5}\times1} \simeq 6.87 \text{ metres}$$

A stone thrown horizontally does not move in circle because it falls down due to gravity and then stops because of friction.

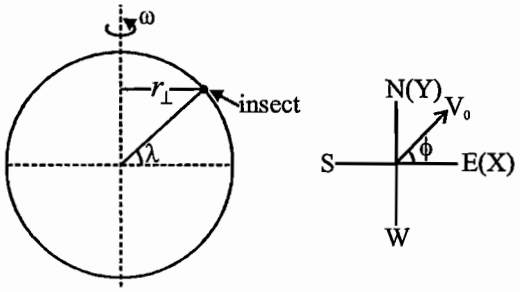
Example: An insect of mass m crawls on earth's surface with velocity v_0 at ϕ angle north of east. The latitude of place where insect is situated is λ. If ω be angular velocity of earth's spinning motion and R be radius of earth. Calculate centrifugal and coriolis force on the insect.

Soln. Velocity of insect w.r.t. earth

$\vec{v}' = v_0(\cos\phi\hat{i} + \sin\phi\hat{j})$

perpendicular distance of insect from axis of rotation is $r_{\perp} = R\cos\lambda$

$\therefore$ centrifugal force $= m\omega^2 r_{\perp} = m\omega^2 R\cos\lambda$



coriolis force $\vec{F}_{cor} = 2m\vec{v}'\times\vec{\omega}$

$= 2mv_0\omega(\cos\phi\hat{i} + \sin\phi\hat{j})\times(\cos\lambda\hat{j} + \sin\lambda\hat{k})$

$= 2mv_0\omega(\cos\phi\cos\lambda\hat{k} - \cos\phi\sin\lambda\hat{j} + \sin\phi\sin\lambda\hat{i})$

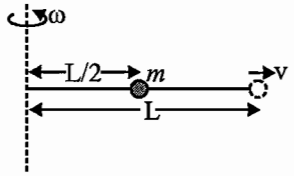
$\therefore$ $\left|\vec{F}_{cor}\right| = 2mv_0\omega\sqrt{\cos^2\phi\cos^2\lambda + \cos^2\phi\sin^2\lambda + \sin^2\phi\sin^2\lambda}$

$= 2mv_0\omega\sqrt{\cos^2\phi + \sin^2\phi.\sin^2\lambda}$

Example: A bead of mass 'm' lies at the centre of a thin wire of length 'L'. The wire is then rotated with uniform angular velocity ω about its one end due to which bead slides away from the axis of rotation. Find the velocity of bead with respect to the rod at the moment when the bead is about to leave the free end of the wire.

Soln. Due to rotation of the wire the bead is acted upon by centrifugal as well as coriolis force (while moving). However it is centrifugal force which accelerates the bead away from the axis of rotation. Let v be the speed of bead w.r.t. the wire when it leaves the other end. Let us apply work energy theorem to find v.

Work done by centrifugal force = change in kinetic energy

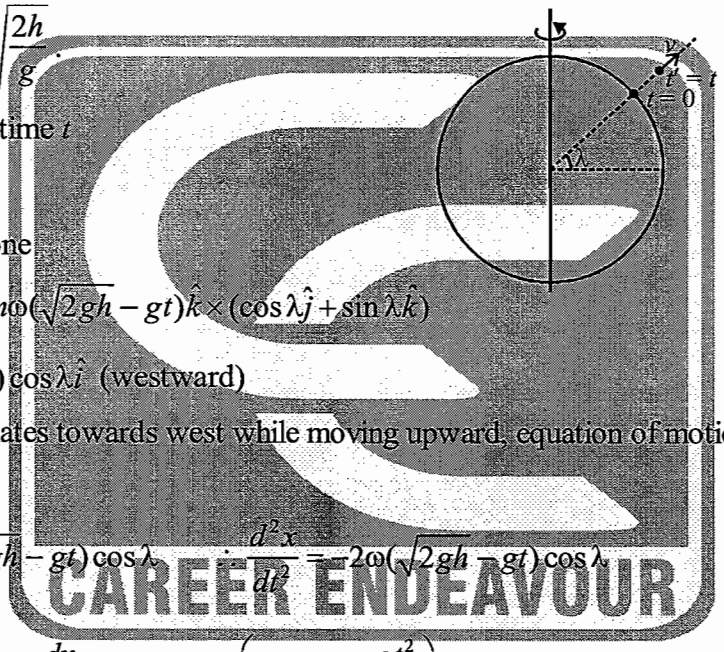


$$\int_{L/2}^{L} m\omega^2 r\,dr = \frac{1}{2}mv^2 - 0, \quad \frac{m\omega^2}{2}\left[L^2 - \frac{L^2}{4}\right] = \frac{1}{2}mv^2$$

$$\therefore v = \frac{\sqrt{3}}{2}\omega L$$

Example: When a stone is dropped from a height 'h' it undergoes an eastward deviation of x_0 on reaching the earth's surface. If a stone is thrown vertically upward such that it reaches a maximum height 'h' then what will be its westward deviation.

Soln. The stone reaches maximum height 'h' therefore its projection speed must be $\sqrt{2gh}$ and time to reach maximum height is $\sqrt{\frac{2h}{g}}$.



Velocity of stone at time t

$$\vec{v}' = (\sqrt{2gh} - gt)\hat{k}$$

Coriolis force on stone

$$\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega} = 2m\omega(\sqrt{2gh} - gt)\hat{k} \times (\cos\lambda\hat{j} + \sin\lambda\hat{k})$$

$$= -2m\omega(\sqrt{2gh} - gt)\cos\lambda\,\hat{i} \quad \text{(westward)}$$

Therefore stone deviates towards west while moving upward, equation of motion along west direction (x-axis)

$$\frac{md^2x}{dt^2} = -2m\omega(\sqrt{2gh} - gt)\cos\lambda \qquad \therefore \frac{d^2x}{dt^2} = -2\omega(\sqrt{2gh} - gt)\cos\lambda$$

CAREER ENDEAVOUR

On integrating we get $\frac{dx}{dt} = -2\omega\cos\lambda\left(\sqrt{2gh}\,t - \frac{gt^2}{2}\right) + c$

Initially, $\frac{dx}{dt} = 0$, at $t = 0$ therefore $c = 0$ $\qquad \therefore \frac{dx}{dt} = -2\omega\cos\lambda\left(\sqrt{2gh}\,t - \frac{gt^2}{2}\right)$

On integration we get $x = -2\omega\cos\lambda\left(\sqrt{2gh}\,\frac{t^2}{2} - \frac{gt^3}{6}\right) + c_1$

at $t = 0$, $x = 0$, therefore $c_1 = 0$

$$\therefore x = -2\omega\cos\lambda\left(\sqrt{2gh}\,\frac{t^2}{2} - \frac{gt^3}{6}\right) \quad \text{or} \quad x = -2\omega\cos\lambda g\left(\sqrt{\frac{2h}{g}}\,\frac{t^2}{2} - \frac{gt^3}{6}\right)$$

Since, total time 't' taken to reach maximum height is $t = \sqrt{\frac{2h}{g}}$ therefore

$$x = -2\omega\cos\lambda g\left(\frac{t^3}{2} - \frac{t^3}{6}\right) \quad \text{or} \quad \boxed{x = -\frac{2}{3}\omega g\cos\lambda t^3}$$

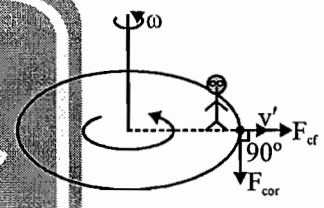
or $x = -\frac{2}{3}\omega g\cos\lambda\left(\frac{2h}{g}\right)^{3/2}$

According to question $x_0 = \frac{1}{3}\omega g\cos\lambda\left(\frac{2h}{g}\right)^{3/2} \quad \therefore |x| = 2x_0$

i.e. westward deviation of a stone is twice the eastward deviation of stone falling from same height.

Example: A person is standing at the edge of a disc of radius R. The disc is rotating about its axis with uniform angular velocity ω. The person throws a stone in radially outward direction with speed $\frac{\omega R}{2}$ relative for the disc. Calculate acceleration of stone as seen by the person soon after throwing. (neglect gravity).

Soln. As seen by the person the stone is acted upon by centrifugal and coriolis forces. The direction of the two forces soon after throwing is shown in the figure.



Net force on the stone is

$$F' = \sqrt{F_{cor}^2 + F_{cf}^2}$$

$$= \sqrt{|2m\vec{v}' \times \vec{\omega}|^2 + (m\omega^2 r_\perp)^2} = \sqrt{\left(2m\frac{\omega R}{2}\omega\sin 90^\circ\right)^2 + (m\omega^2 R)^2}$$

$$F' = \sqrt{2}m\omega^2 R$$

Therefore acceleration of stone as seen by the person is

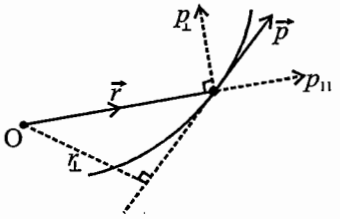
$$a' = \frac{F'}{m} = \sqrt{2}\omega^2 R$$

CAREER ENDEAVOUR

Angular momentum and torque

"Give me a place to stand and with a lever I will move the whole world." **Archimedes**

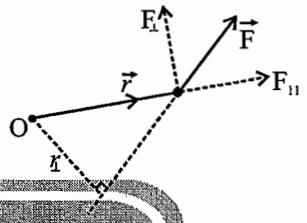
Angular momentum: If $\vec{p}$ be momentum of a particle and $\vec{r}$ be its position vector w.r.t. a point 'o' then angular momentum about point 'o' is defined as



$\vec{L} = \vec{r} \times \vec{p}$

Magnitude of angular momentum is written as $L = rp_\perp$ or $L = r_\perp p$

Torque: If $\vec{F}$ be force acting at some point and $\vec{r}$ be the position vector of the point w.r.t. some origin 'O' then torque about 'O' is defined as $\vec{\tau} = \vec{r} \times \vec{F}$



Magnitude of force is written as $\tau = rF_\perp$ or $\tau = r_\perp F$

If a system is in translational equilibrium then net force is zero however if system is in rotational equilibrium then net torque is zero about any point.

Relation between torque and angular momentum:

$\vec{L} = \vec{r} \times \vec{p}$

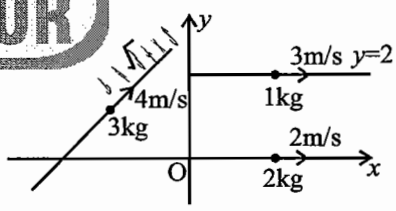
Differentiate w.r.t. time to get $\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}}$

If $\vec{\tau} = 0$ then $\vec{L}$ = constant. This is called conservation of angular momentum.

Example: Three particles are moving in x-y plane with constant velocities as shown in the figure. Calculate angular momentum of each particle about the origin.

Soln. Angular momentum can be calculated in various ways.

Let us use $L = r_\perp p$



For 1 kg particle:

momentum $(p) = mv = 1 \times 3 = 3\,\text{kg m/s}$

perpendicular distance of line of momentum from origin $(r_\perp) = 2m$

$\therefore L = r_\perp p = 2 \times 3 = 6\,\text{kg m}^2/\text{s}$

For 2kg particle: Line of momentum of this particle passes through origin therefore $r_\perp = 0$ $\therefore L = 0$

For 3kg particle: $r_\perp = \dfrac{|y - \sqrt{3}x - 5|_{x=0,y=0}}{\sqrt{1^2 + (\sqrt{3})^2}} = \dfrac{|0 - \sqrt{3} \times 0 - 5|}{\sqrt{4}} = \dfrac{5}{2}$

$p = mv = 3 \times 4 = 12\,\text{kg m/s}$

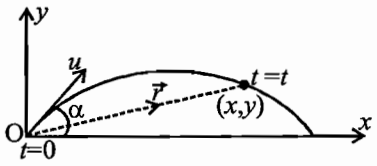
$\therefore \text{L} = r_\perp p = \dfrac{5}{2} \times 12 = 30\,\text{kg m}^2/\text{sec}$

CAREER ENDEAVOUR

Example: A particle of mass m is thrown with initial speed u at angle α with horizontal. If maximum height reached is much small compared to radius of earth. Calculate angular momentum of particle about point of projection after time t.

Soln. Position vector of particle after time t

$\vec{r} = x\hat{i} + y\hat{j}$



$\vec{r} = u\cos\alpha t\hat{i} + (u\sin\alpha t - \frac{1}{2}gt^2)\hat{j}$

velocity of particle after time t

$\vec{v} = \frac{d\vec{r}}{dt} = u\cos\alpha\hat{i} + (u\sin\alpha - gt)\hat{j}$

$$\therefore \vec{L} = \vec{r}\times\vec{p} = m\vec{r}\times\vec{v} = m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u\cos\alpha t & u\sin\alpha t - \frac{1}{2}gt^2 & 0 \\ u\cos\alpha & u\sin\alpha - gt & 0 \end{vmatrix}$$

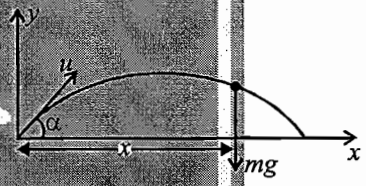
$$= \hat{k}[u^2\cos\alpha\sin\alpha t - gt^2 u\cos\alpha - u^2\cos\alpha\sin\alpha t + \frac{1}{2}gut\cos\alpha t^2] = -\frac{1}{2}gt^2 u\cos\alpha\hat{k}$$

Another method: At time t, torque on the particle about point of projection is

$\tau = r_\perp F = xmg$

or $\vec{\tau} = -mgx\hat{k}$

$\vec{\tau} = -mgu\cos\alpha t\hat{k}$



We know that $\vec{\tau} = \frac{d\vec{L}}{dt}$

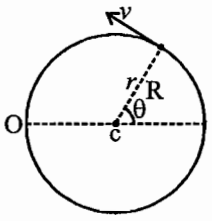
$$\therefore \frac{d\vec{L}}{dt} = -mgu\cos\alpha t\hat{k} \quad \text{or} \quad \int_0^{\vec{L}} d\vec{L} = -mgu\cos\alpha\int_0^t t\,dt\hat{k}$$

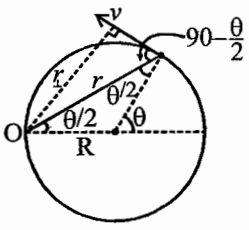
$$\therefore \vec{L} = -\frac{1}{2}mgut^2\cos\alpha\hat{k}$$

Example: A particle of mass 'm' moves with constant speed v along a circle of radius R. Calculate angular momentum about centre of circle. Also calculate angular momentum about a point on the circumference of circle.

Soln. About centre $L_c = r_\perp . p = Rmv$

About a point on circumference



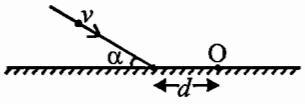


$L_O = r_\perp p = r\sin\left(90 - \frac{\theta}{2}\right)mv$ (from figure)

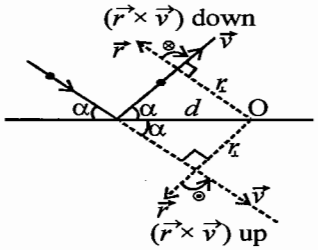
$= r\cos\frac{\theta}{2}mv = \left(2R\cos\frac{\theta}{2}\right)\cos\frac{\theta}{2}mv$ (use sine rule to get $r = 2R\cos\frac{\theta}{2}$)

$= 2Rmv\cos^2\frac{\theta}{2}$

Example: A particle of mass m moving with constant speed v strikes a floor elastically. Calculate change in angular momentum about a point which is at a distance 'd' from the point of impact as shown in figure.



Soln. The collision is elastic therefore particle will rebound with same speed and at same angle with the floor as shown in the figure.



$\therefore r_\perp = d\sin\alpha$

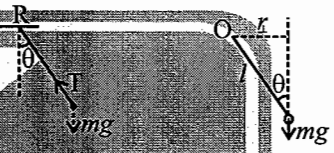
Angular momentum before impact

$L_1 = r_\perp p = d\sin\alpha\, mv, \quad L_2 = r_\perp p = -d\sin\alpha\, mv$

$\therefore$ Change in angular momentum $\Delta L = L_2 - L_1 = -2mvd\sin\alpha$

Example: A simple pendulum consisting of a small bob of mass m and light string of length 'l' oscillates in vertical plane. Calculate torque due to gravity and tension in the string at the instant string makes θ angle with downward vertical.

Soln. Line of action of tension passes through point of suspension. Therefore $r_\perp = 0$



Thus, torque due to tension $= 0$

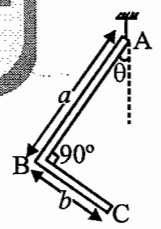
For mg, $r_\perp = l\sin\theta$

$\therefore \tau = r_\perp F$

$= l\sin\theta mg = mgl\sin\theta$

Example: An L-shaped uniform wire is suspended from a string as shown in figure. If length of the two sections of wire be 'a' and 'b' then what is value of angle θ made by upper section with vertical.

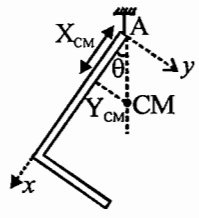
Soln. The wire is in rotational equilibrium therefore centre of mass of the rod must lie on the vertical line through point A so that gravity does not produce any torque about point A.



To find the position of centre of mass let us choose coordinate system as shown in figure.

Let λ be mass per unit length of the wire.

$$\therefore X_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{\lambda a.\frac{a}{2} + \lambda ba}{\lambda a + \lambda b} = \frac{a(a+2b)}{2(a+b)}$$



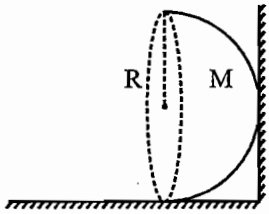
$$Y_{CM} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2} = \frac{\lambda a.0 + \lambda b.\frac{b}{2}}{\lambda a + \lambda b} = \frac{b^2}{2(a+b)}$$

$$\therefore \tan\theta = \frac{Y_{CM}}{X_{CM}} = \frac{b^2}{a(a+2b)}$$

or $\theta = \tan^{-1}\left(\frac{b^2}{a(a+2b)}\right)$ if $a = b$ then $\theta = \tan^{-1}\left(\frac{1}{3}\right)$

CAREER ENDEAVOUR

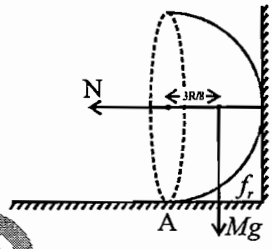
Example: A solid hemisphere of mass M and radius R is placed against a smooth wall as shown in the figure. What is normal reaction on the sphere due to the wall.



Soln: Centre of mass of the hemisphere lies at a distance $\frac{3R}{8}$ from its base centre. The sphere is in equilibrium therefore torque about point A must be zero.

$$\therefore NR - Mg.\frac{3R}{8} = 0$$

or, $N = \frac{3Mg}{8}$



Example: A thin rod of mass M and length L is suspended from one end while its other end lies on a smooth horizontal plane and makes an angle α with the plane. Find the normal reaction from the plane and tension in the string from which it is suspended.

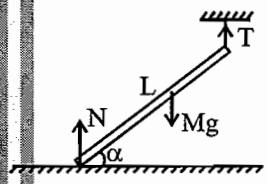
For translational equilibrium $T + N = Mg$ ... (a)

For rotational equilibrium

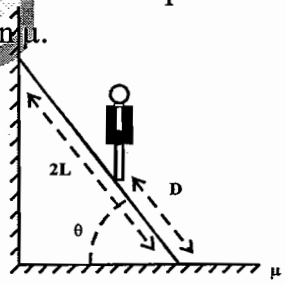
Torque about lower end of the rod is zero

$$\therefore T.L\cos\alpha - Mg\frac{L}{2}\cos\alpha = 0 \quad \therefore \quad T = \frac{Mg}{2}$$

from (a) $N = Mg - T = Mg - \frac{Mg}{2} = \frac{Mg}{2}$



Example: A uniform ladder of length 2L and mass 'm' leans against a wall in a vertical plane at an angle θ to the horizontal. The floor is rough, having a coefficient of static friction μ. A person of mass M stands on the ladder at a distance D from its base (see figure). If the wall is frictionless, the maximum distance (D_{max}) up the ladder that the person can reach before the ladder slips is



Soln. Let us take ladder plus person as system. Various forces acting on the system are shown in the figure. For translational equilibrium we must have

$N_1 = (M + m)g$ ... (a)

$N_2 = f_r$ ... (b)

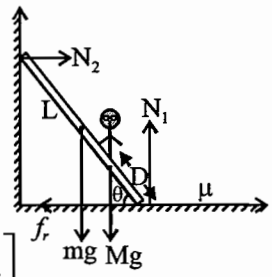
For rotational equilibrium torque about lower end of the ladder must be zero. Therefore,

$MgD\cos\theta + mgL\cos\theta - N_2.2L\sin\theta = 0$

$$\therefore N_2 = \frac{g\cot\theta}{2}\left[\frac{MD}{L} + m\right]$$

CAREER ENDEAVOUR

$\therefore$ from (b) we get, $f_r = \frac{g\cot\theta}{2}\left[\frac{D}{L}M + m\right]$



since $f_r \le \mu N_1$, $\therefore \frac{g\cot\theta}{2}\left[\frac{D}{L}M + m\right] \le \mu(M+m)g$

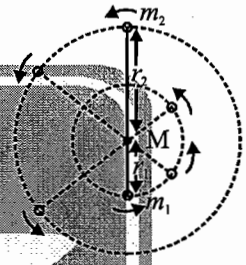
or $D \le \frac{L}{M}[2\mu(M+m)\tan\theta - m]$ $\therefore D_{max} = L\left[2\mu\left(1+\frac{m}{M}\right)\tan\theta - \frac{m}{M}\right]$

Double star problem: A double star consists of two stars revolving in circular orbits about their common centre of mass. The necessary centripetal force for circular motion is provided by their mutual gravitational attraction.

Let us consider a double star which consists of two star of masses m_1 and m_2 separated by a distance r.

There is no external force on the system therefore centre of mass of the system does not move.

Let r_1 and r_2 be radii of the orbits of the two stars.



$\therefore$ $r_1 = \frac{m_2 r}{m_1 + m_2} \Rightarrow \frac{r_1}{r_2} = \frac{m_2}{m_1}$,

$r_2 = \frac{m_1 r}{m_1 + m_2}$

Since centre of mass is stationary. Therefore, the angular velocity of the two stars must be same.

$\therefore$ For star 1 $m_1\omega^2 r_1 = \frac{Gm_1 m_2}{r^2} \Rightarrow \omega = \sqrt{\frac{G(m_1+m_2)}{r^3}}$

For start 2 $m_2\omega^2 r_2 = \frac{Gm_1 m_2}{r^2} \Rightarrow \omega = \sqrt{\frac{G(m_1+m_2)}{r^3}}$

Ratio of linear speeds of two stars

$\frac{v_1}{v_2} = \frac{\omega r_1}{\omega r_2} = \frac{r_1}{r_2} = \frac{m_2}{m_1}$ $\therefore \boxed{\frac{v_1}{v_2} = \frac{m_2}{m_1}}$

Ratio of kinetic energy of two stars

$\frac{K_1}{K_2} = \frac{\frac{1}{2}m_1 v_1^2}{\frac{1}{2}m_2 v_2^2} = \frac{m_1(\omega r_1)^2}{m_2(\omega r_2)^2} = \frac{m_1}{m_2}\cdot\left(\frac{r_1}{r_2}\right)^2 = \frac{m_2}{m_1}$ $\therefore \boxed{\frac{K_1}{K_2} = \frac{m_2}{m_1}}$

Ratio of angular momentum of two stars

$\frac{L_1}{L_2} = \frac{m_1 v_1 r_1}{m_2 v_2 r_2} = \frac{m_1 \omega r_1^2}{m_2 \omega r_2^2} = \frac{m_2}{m_1}$ $\therefore \boxed{\frac{L_1}{L_2} = \frac{m_2}{m_1}}$

ratio of magnitude of linear momentum

$\frac{P_1}{P_2} = \frac{m_1 v_1}{m_2 v_2} = \frac{m_1 \omega r_1}{m_2 \omega r_2} = \frac{m_1}{m_2}\cdot\frac{r_1}{r_2} = \frac{m_1}{m_2}\cdot\frac{m_2}{m_1}$ $\therefore \boxed{\frac{P_1}{P_2} = 1}$

CAREER ENDEAVOUR

Total energy of double star system

$$E = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \frac{Gm_1.m_2}{r} = \frac{1}{2}m_1\omega^2r_1^2 + \frac{1}{2}m_2\omega^2r_2^2 - \frac{Gm_1.m_2}{r}$$

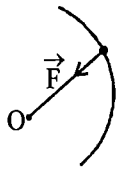
put values of r_1, r_2 and ω to get

$$E = \frac{1}{2}\frac{G(m_1+m_2)}{2r^3}\left[\frac{m_1m_2^2r^2 + m_1^2r^2m_2}{(m_1+m_2)^2}\right] - \frac{Gm_1m_2}{r}$$

$$= \frac{Gm_1.m_2}{2r} - \frac{Gm_1.m_2}{r} \qquad \boxed{E = \frac{-Gm_1.m_2}{2r}}$$

Central force motion

A central force is expressed as $\vec{F} = \pm f(r)\hat{r}$, positive sign is used for repulsive and negative sign for attractive force. Central force acting on a particle is always directed towards a point which is called centre of force.



Features of central force motion:

- Central forces are conservative, therefore total energy of a particle moving under central force is conserved.
- Torque about centre of force is zero, therefore angular momentum about centre of force is conserved. This implies that central force motion is a planar motion.
- Areal velocity is constant, that is area swept by line joining the particle to the centre of force per unit time is constant.

$$\frac{\Delta A}{\Delta t} = \text{constant}$$

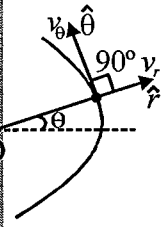
Relation between angular momentum and areal velocity: Central force motion is a planar motion, therefore we can use plane polar coordinate to describe dynamics of particle.

Therefore velocity of particle is $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

Angular momentum of particle about centre of force 'O' is

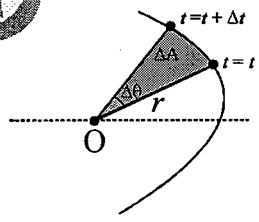
$L = mv_\theta r = mr\dot{\theta}r = mr^2\dot{\theta}$

$$\boxed{L = mr^2\dot{\theta}}$$



Areal velocity: $\frac{\Delta A}{\Delta t} = \frac{\frac{1}{2}r^2\Delta\theta}{\Delta t} = \frac{1}{2}r^2\frac{\Delta\theta}{\Delta t} = \frac{1}{2}r^2\dot{\theta}$

CAREER ENDEAVOUR



from expression of angular momentum $r^2\dot{\theta} = \frac{1}{m}$

$$\therefore \boxed{\text{Areal velocity} = \frac{L}{2m}}$$

Differential equation of orbit: Acceleration of a particle moving in a plane can be expressed in plane polar coordinate as

$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$

Therefore in central force motion Newton's equation is

$\vec{F} = m\vec{a}$, therefore,

$f(r)\hat{r} = m[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}]$

Therefore, equating coefficient of $\hat{r}$ and $\hat{\theta}$ we get

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \qquad \text{... (i)}$$

and $m(\ddot{r} - r\dot{\theta}^2) = f(r)$ ... (ii)

Equation (i) can also be obtained by differentiating $L = mr^2\dot{\theta}$ therefore it implies constancy of angular momentum. Equation (ii) is differential equation for the dynamics of particle under central force. To obtain differential equation of orbit we must remove time from equation (ii). To accomplish this we introduce a new variable $u = 1/r$ and use $\dot{\theta} = L/mr^2 = Lu^2/m$. This leads to,

$$\boxed{\frac{\partial^2 u}{\partial \theta^2} + u = -\frac{mf(r)}{L^2u^2}} \quad \text{... (iii)}$$

equation (iii) is differential equation of orbit. The above equation is mostly used for following purposes.

(a) To find $f(r)$ when equation of orbit is given. That is a relation between r and θ is given then $f(r)$ can be found.

(b) If $f(r)$ is given then to find equation of orbit.

Example: Equation of the orbit of a particle moving under central force is $r\theta = \beta$, where β is a constant. Find the force acting on the particle.

Soln. $r\theta = \beta$, therefore, $u = \frac{1}{r} = \frac{\theta}{\beta}$ $\therefore \frac{\partial^2 u}{\partial \theta^2} = 0$

Differential equation of orbit, $\frac{\partial^2 u}{\partial \theta^2} + u = \frac{-mf(r)}{L^2u^2}$

$\therefore \; 0 + u = \frac{-mf(r)}{L^2u^2}$ $\therefore f(r) = \frac{-L^2u^3}{m} = \frac{-L^2}{mr^3}$

Example: Equation of orbit of a particle moving under central force is $r^n = a\cos n\theta$. Find the force on the particle.

Soln. Given, $r^n = a\cos n\theta$, therefore, $u^n = \frac{1}{a\cos n\theta} = \frac{1}{a}\sec n\theta$ ... (a)

CAREER ENDEAVOUR

Taking ln both sides we get, $n\ln u = \ln\left(\frac{1}{a}\right) + \ln\sec n\theta$

Differentiating w.r.t. θ we get, $\frac{n}{u}\frac{\partial u}{\partial \theta} = n\tan n\theta$ $\therefore \frac{\partial u}{\partial \theta} = u\tan n\theta$

Differentiating again w.r.t. θ we get $\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial u}{\partial \theta}\tan n\theta + u\,n\sec^2 n\theta = u\tan^2 n\theta + u\,n\sec^2 n\theta$

Differential equation of orbit is $\frac{\partial^2 u}{\partial \theta^2} + u = \frac{-mf(r)}{L^2u^2}$

$u\tan^2 n\theta + u\,n\sec^2 n\theta + u = \frac{-mf(r)}{L^2u^2}$, $u\sec^2 n\theta + u\,n\sec^2 n\theta = \frac{-mf(r)}{L^2u^2}$

$\therefore f(r) = -\frac{L^2u^3(1+n)\sec^2 n\theta}{m}$

from (a) $\sec^2 n\theta = a^2u^{2n}$

$$\therefore \quad f(r) = \frac{-L^2(n+1)a^2u^{2n+3}}{m}$$

$$\therefore \quad f(r) = \frac{-L^2(n+1)a^2}{m}\cdot\frac{1}{r^{2n+3}}$$

or $$\boxed{f(r) \propto \frac{1}{r^{2n+3}}}$$

Example: A particle of mass m is moving under a central force. The angular momentum of the particle is L and its equation of orbit is $r = Ae^{k\theta}$. Calculate potential energy of the particle.

Soln. Potential energy is given as $V(r) = -\int f(r)dr$ ... (a)

Therefore, let us first find $f(r)$.

Given $r = Ae^{k\theta}$, therefore, $u = \frac{1}{A}e^{-k\theta}$ $\quad \therefore \frac{\partial^2 u}{\partial\theta^2} = k^2u$

Differential equation of orbit is $\frac{\partial^2 u}{\partial\theta^2} + u = \frac{-m f(r)}{L^2u^2}$ $\quad \therefore (k^2+1)u = \frac{-m f(r)}{L^2u^2}$

$$\therefore f(r) = \frac{-L^2(k^2+1)u^3}{m} \quad \text{or} \quad f(r) = \frac{-L^2(k^2+1)}{mr^3}$$

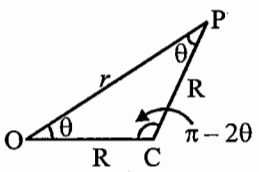
Therefore, from (a) $V(r) = -\int f(r)dr = \frac{-L^2(k^2+1)}{2mr^2}$

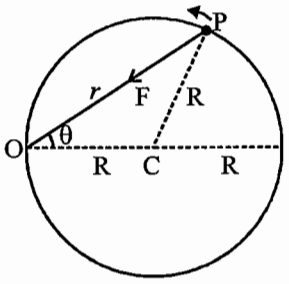
Example: A particle is moving in a circle under a force which is always directed towards a point on the circumference of the circle. Find the force law.

Soln. Let R be the radius of the circle and r be distance of prticle from the centre of force. Let θ be the angular coordinate of the particle as shown in the figure.

Therefore using sine rule we get,

$$\frac{r}{\sin(\pi-2\theta)} = \frac{R}{\sin\theta} \quad \therefore r = 2R\cos\theta$$





In previous example we have seen that for $r^n = a\cos n\theta$, $F \propto \frac{1}{r^{2n+1}}$. In present case $n = 1$, therefore, force law is $F \propto \frac{1}{r^5}$.

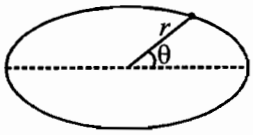
CAREER ENDEAVOUR

Polar equation of orbit under $F=-\frac{k}{r^2}$ force: Potential energy corresponding to this force is $V(r)=-\int F dr=-\frac{k}{r}$. Let L be the angular momentum of particle and E be total energy. Therefore,

$L=mr^2\dot{\theta}$ and $E=\frac{1}{2}mv^2=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)=\frac{1}{2}m\dot{r}^2+\frac{L^2}{2mr^2}-\frac{k}{r}$

Differential equation of orbit is $\frac{\partial^2 u}{\partial\theta^2}+u=\frac{-m f(r)}{L^2u^2}$

Using $f(r)=-\frac{k}{r^2}=-ku^2$ we get $\frac{\partial^2 u}{\partial\theta^2}+u=+\frac{mk}{L^2}$



Solution of above differential equation gives $\boxed{r=\frac{L^2/mk}{1+\sqrt{1+2E\frac{L^2}{mk^2}}\cos\theta}}$

This is polar equation of orbit under $F=-\frac{k}{r^2}$ force. The above is rewritten as,

$\boxed{r=\frac{l}{1+e\cos\theta}}$ where $l=\frac{L^2}{mk}$ = Latus rectum of the conic section (curve) represented by above equation and $e=\sqrt{1+2E\frac{L^2}{mk^2}}$ is eccentricity of the orbit.

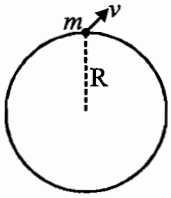
CAREER ENDEAVOUR

Classification of orbits: From polar equation of orbit we see that under a given force law orbit of a particle depends on its energy. Therefore, in present case,

Energy	Eccentricity	Orbit
$\frac{-mk^2}{2L^2}<E<0$	e < 1	Elliptical
$E=0$	e = 1	Parabola
$E>0$	e > 1	Hyperbola
$E=\frac{-mk^2}{2L^2}$	e = 0	Circle

Therefore if nature of orbit is to be found for $F=-\frac{k}{r^2}$ we can just check if its energy satisfies any of the above energy conditions.

Example: A particle is thrown from earth's surface with speed $v=\sqrt{\frac{1.5GM}{R}}$, where M and R are mass and radius of the earth respectively. What is the nature of the path of the particle.



Soln. Force on the particle is $F=\frac{-GMm}{r^2}$

Therefore potential energy of the particle is, $V=-\int F dr=\frac{-GMm}{r}$

Near earth's surface $V = \dfrac{-GMm}{R}$

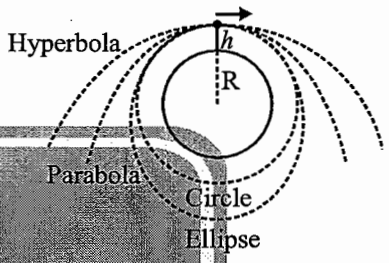
Total energy of the particle is $E = \dfrac{1}{2}mv^2 - \dfrac{GMm}{R} = \dfrac{1}{2}m.\left(\dfrac{1.5GM}{R}\right) - \dfrac{GMm}{R} = \dfrac{-GMm}{4R} < 0$

Since $E < 0$, therefore path of the particle will be elliptical.

Nature of orbit from velocity of projectile: In the previous example we have seen that the orbit of projectile depends on the initial velocity with which it is thrown. In the following we give nature of orbit for given initial velocity.

Suppose a particle is thrown from some height above the earth's surface as shown in figure.

Velocity	**Path**
$\sqrt{\dfrac{GM}{R+h}}$	Circle
$\sqrt{\dfrac{GM}{R+h}} <$ speed $< \sqrt{\dfrac{2GM}{R+h}}$	Ellipse
$\sqrt{\dfrac{2GM}{R+h}}$	Parabola
Speed $> \sqrt{\dfrac{2GM}{R+h}}$	Hyperbola



Example: For a particle moving under gravitational force pericentre distance in parabolic orbit is r_p while the radius of the circular orbit with same angular momentum is r_c. What is relation between r_p and r_c?

Soln. Pericentre distance is the minimum distance ($\cos\theta$ = max = 1) and for parabolic orbit $e = 1$.

Therefore, $r_{min} = \dfrac{l}{1+e\cos\theta} = \dfrac{l}{2} = r_p$

for circular orbit $e = 0$, therefore $r_c = \dfrac{l}{1+e\cos\theta} = l \quad \therefore r_p = \dfrac{r_c}{2}$

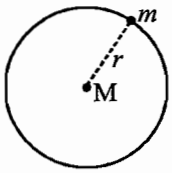
CAREER ENDEAVOUR

Energy and angular momentum of planet: Let m be the mass of planet and M be the mass of the sun. Let us assume that sun is stationary and planet revolves around the sun. Here we will use polar equation of orbit to get the expression.

Circular orbit: Let the orbit is circular and its radius is r, eccentricity of circular orbit is zero.

$$\therefore \quad e = \sqrt{1 + 2E.\frac{L^2}{k^2}} = 0$$

$$\therefore \quad E = \frac{-mk^2}{2L^2} \qquad \text{... (i)}$$



From polar equation of orbit $r = \dfrac{L^2/mk}{1+0}$

$$\therefore \quad L^2 = mkr \qquad \text{... (ii)}$$

for gravitational force $k = GMm$, $L = \sqrt{Gm^2Mr}$

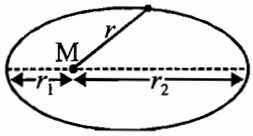
from (i), $E = \dfrac{-m(GMm)^2}{2.Gm^2Mr} = \dfrac{-GMm}{2r}$

if v be the orbital speed then $L = mvr \Rightarrow v = \dfrac{L}{mr} = \sqrt{\dfrac{GM}{r}}$

Therefore, for the planet in circular orbit angular momentum energy and speed are given as

and $\boxed{L = m\sqrt{GMr},\ E = -\dfrac{GMm}{2r},\ v = \sqrt{\dfrac{GM}{r}}}$

Elliptical orbit: Let r_1 and r_2 be minimum and maximum distance of planet from the sun



From polar equation of orbit $r = \dfrac{L^2/mk}{1+\sqrt{1+\dfrac{2E.L^2}{mk^2}}\cos\theta}$

for minimum distance $\cos\theta = 1$

$\therefore\ r_1 = \dfrac{L^2/mk}{1+\sqrt{1+\dfrac{2E.L^2}{mk^2}}}$ or $1+\sqrt{1+\dfrac{2E.L^2}{mk^2}} = \dfrac{L^2}{mkr_1}$ ... (a)

Similarly for maximum distance $(\cos\theta = -1)$

$1-\sqrt{1+\dfrac{2E.L^2}{mk^2}} = \dfrac{L^2}{mkr_2}$ ... (b)

Add equations (a) and (b) and use $k = GMm$ to get $L = m\sqrt{\dfrac{2GMr_1.r_2}{r_1+r_2}}$

CAREER ENDEAVOUR

Put value of L in any of the two equations to get $E = -\dfrac{GMm}{r_1+r_2}$

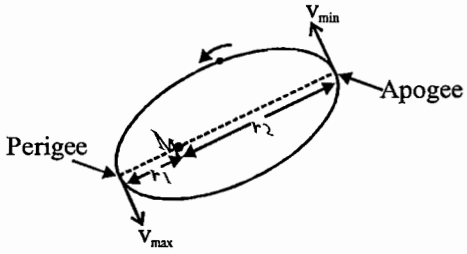
Therefore, for a planet in elliptical orbit energy and angular momentum are given as

$$\boxed{E = -\frac{GMm}{r_1+r_2},\ L = m\sqrt{\frac{2GMr_1.r_2}{r_1+r_2}}}$$

Maximum and minimum velocity in elliptical orbit: Angular momentum of planet is conserved therefore its velocity is maximum at perigee and minimum at apogee.

Therefore, $mv_{max}.r_{min} = L$ or $v_{max} = \dfrac{L}{mr_{min}} = \dfrac{L}{mr_1}$

Using value $L = m\sqrt{\dfrac{2GMr_1.r_2}{r_1+r_2}}$ we get $v_{max} = \sqrt{\dfrac{2GM\,r_2}{r_1(r_1+r_2)}}$

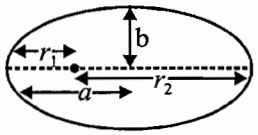


Similarly $m v_{min} . r_{max} = L$ gives $v_{min} = \frac{L}{m r_2} = \sqrt{\frac{2GMr_1}{r_2(r_1+r_2)}}$

Therefore, $\boxed{v_{max} = \sqrt{\frac{2GMr_2}{r_1(r_1+r_2)}},\ v_{min} = \sqrt{\frac{2GMr_1}{r_2(r_1+r_2)}}}$

E, L, v_{max}, v_{min} in terms of ellipse parameters: Let a, b and e be the semi-major axis, semi-major axis and eccentricity of ellipse.

$\therefore\ r_1 = a(1-e),\ r_2 = a(1+e),\ e = \sqrt{1-\frac{b^2}{a^2}}$



Using result of previous section we get

$E = \frac{-GMm}{2a}$ (E depends only on length of major axis)

$L = m\sqrt{GMa(1-e^2)}$, $v_{max} = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}$, $v_{min} = \sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}$

Example: Ratio of maximum to minimum speed of a planet revolving around the sun in an elliptical orbit is 2:1, What is eccentricity of the orbit?

Soln. Given $\frac{v_{max}}{v_{min}} = \frac{2}{1}$ $\therefore \frac{\sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}}{\sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}} = \frac{2}{1}$ or $\frac{1+e}{1-e} = \frac{2}{1}$ $\therefore e = \frac{1}{3}$

Example: An object is thrown from the earth's surface with initial speed $\sqrt{\frac{1.5GM}{R}}$ due to which it moves in an elliptical orbit. What is the semi-major axis of the ellipse.

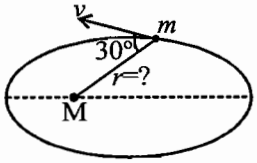
Soln. Total energy of the particle

$$E = \frac{-GMm}{R} + \frac{1}{2}mv^2 = \frac{-GMm}{R} + \frac{1}{2}m.\frac{1.5GM}{R} = \frac{-GMm}{4R}$$

In central force motion energy of the particle is conserved. Therefore its energy in elliptical orbit will have the same value. Let a be semi-major axis of the ellipse.

Therefore, $E = \frac{-GMm}{2a} = \frac{-GMm}{4R}$ $\therefore a = 2R$

Example: A planet is revolving around the sun in elliptical orbit of semi-major axis 'a' and eccentricity $\sqrt{3}/2$. At certain instant of time velocity vector of planet makes 30° with the radial line. What is distance of planet from the sun at this instant.



Soln. Energy of the planet is

$$E = \frac{-GMm}{r} + \frac{1}{2}mv^2 \qquad \text{... (i)}$$

Angular momentum of the planet is

$$L = mvr\sin 30° = \frac{mvr}{2} \qquad \text{... (ii)}$$

We know that $L = m\sqrt{GMa(1-e^2)}$, $e = \frac{\sqrt{3}}{2}$

$$\therefore \quad \frac{mvr}{2} = m\sqrt{GMa\frac{1}{4}} \quad \text{or} \quad v = \frac{\sqrt{GMa}}{r}$$

We know that $E = \frac{-GMm}{2a}$.

Therefore from (i) we get, $\frac{-GMm}{2a} = \frac{-GMm}{r} + \frac{1}{2}m.\frac{GMa}{r^2}$

or $-\frac{1}{2a} = -\frac{1}{r} + \frac{a}{2r^2}$ or $r^2 - 2ar + a^2 = 0 \quad \therefore r = a$

Example: A planet is revolving around the sun in a circular orbit. Due to some reason the speed of the planet suddenly becomes double. What is new orbit of the planet.

Soln. Orbital speed of the planet is $\sqrt{\frac{GM}{r}}$

New speed of the planet $= 2\sqrt{\frac{GM}{r}}$

Therefore, new energy of the planet $= \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}m.\frac{4GM}{r} - \frac{GMm}{r} = \frac{2GMm}{r} > 0$

Total energy of the planet becomes positive on doubling its speed therefore new orbit of the planet will be hyperbolic.

CAREER ENDEAVOUR

Effective potential (Reducing 2-d problem into 1-d problem): Under central force, a particle has planar motion. That is motion of the particle is 2-dimensional. Therefore total energy is written as $E = \frac{1}{2}mv^2 + V(r)$

or, $E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$

Angular momentum of the particle is constant of motion and it is given as, $L = mr^2\dot{\theta}$

$$\therefore \quad \dot{\theta} = \frac{L}{mr^2}$$

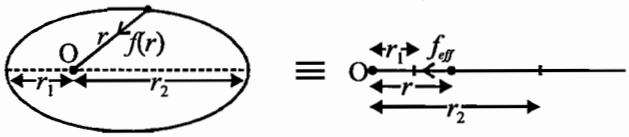
Let us put the value of $\dot{\theta}$ in the energy expression to get $E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r)$

or $\boxed{E = \frac{1}{2}m\dot{r}^2 + V_{eff}(r)}$ ... (a)

where $\boxed{V_{eff}(r) = \frac{L^2}{2mr^2} + V(r)}$

Equation (a) can be interpreted as follows.

Two dimensional motion under a central potential $V(r)$ is equivalent to radial motion under effective potential $V_{eff}(r)$.

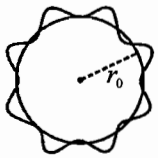


$$f = \frac{-\partial V}{\partial r} \qquad f_{eff} = \frac{-\partial V_{eff}}{\partial r}$$

Therefore, reason for defining an effective potential is that it is easier to discuss 1-d problem than 2-d problem. Before we start dealing with problem let us first understand how the interpretation in one dimension is different from that in two dimension.

Actual 2-D problem

- Energy $E = \frac{1}{2}mv^2 + V(r)$
- At turning points velocity is $\perp$ to the radial line
- Circular motion under a central force is uniform motion and $f(r) = \frac{mv^2}{r}$
- Oscillation about stable radius is shown as



Equivalent 1-D problem

- Energy $E = \frac{1}{2}m\dot{r}^2 + V_{eff}(r)$
- At turning points $\dot{r} = 0$. Therefore it appears as if particle stops there.
- In 1-d description of circular motion particle appears stationary, $\dot{r} = 0$ and $f_{eff} = 0$.
- An equivalent oscillation in 1–d is linear oscillation

CAREER ENDEAVOUR

Finding the nature of possible orbits under a given central force / potential: In previous section we have seen that various types of orbits are possible under $F = -k/r^2$ force and the nature of orbit depends on energy of particle. It is tedious task to find polar equation of orbit for various forces and then conclude about the nature of orbit for a given energy as we did in previous section. Therefore instead of discussing the problem in 2-d we discuss equivalent problem in 1-d.

In equivalent 1-d problem total energy is written as $E = \frac{1}{2}m\dot{r}^2 + V_{eff}$

or $\frac{1}{2}m\dot{r}^2 = E - V_{eff}$

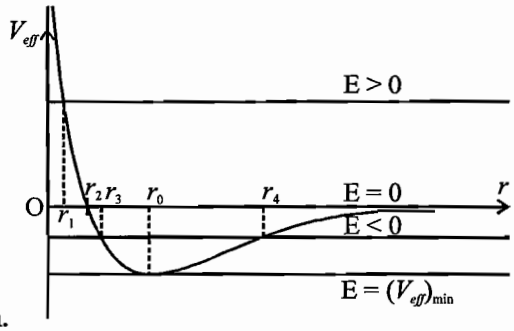
Therefore, E must be greater or equal to V_{eff}. Thus we conclude that if we plot V_{eff} versus r, movement of particle will be limited to that region only where V_{eff} line lies below the total energy line. That is particle cannot go in the region where V_{eff} line is above the total energy (E) line.

Dynamics under Inverse square force $F = -k/r^2$ or $V(r) = -k/r$:

Let m be mass of particle and L be its angular momentum.

Therefore, $V_{eff}(r) = V(r) + \frac{L^2}{2mr^2} = \frac{-k}{r} + \frac{L^2}{2mr^2}$

The plot of $V_{eff}(r)$ versus r is shown in the figure below. To plot the graph consider the following



If $f(r) = ar^m - br^n$ where $m > n$ then

for $r \to 0$ lower power term (br^n) dominates

for $r \to \infty$ higher power term (ar^m) dominates

From the plot of V_{eff} versus r we draw following conclusion.

(i) There is a minimum at $r = r_0$. Therefore if particle maintains its distance from centre of force to be equal to r_0 then its orbit is stable. i.e. if particle moves in a circle of radius r_0 then its orbit is stable.

Therefore to find radius (r_0) of stable orbit we use $\left.\frac{dV_{eff}}{dr}\right|_{r=r_0} = 0$

(ii) For E > 0 minimum distance of the particle from centre of force is r_1 and maximum distance is infinity that is $r_1 \le r \le \infty$. Therefore, for E > 0 orbit is unbounded.

(iii) For E = 0, $r_{min} = r_2$, $r_{max} = \infty$, therefore $r_1 \le r \le \infty$ and orbit is unbounded.

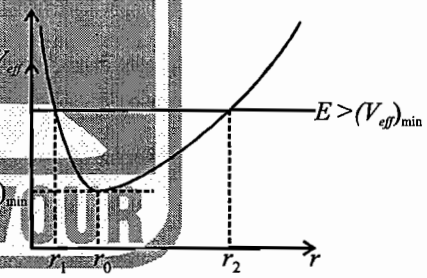
(iv) For E < 0, $r_{min} = r_3$, $r_{max} = r_4$, therefore $r_3 \le r \le r_4$ orbit is bounded.

(v) $E = (V_{eff})_{min}$, only $r = r_0$ is possible, therefore orbit is a circle of radius r_0.

(vi) $E < (V_{eff})_{min}$ is not allowed

Dynamics under Hookes force: $F = -kr$, $V(r) = \frac{1}{2}kr^2$:

$$V_{eff} = \frac{1}{2}kr^2 + \frac{L^2}{2mr^2}$$



Plot of V_{eff} versus r is shown in the figure and from graph we draw following conclusion.

(i) For $E = (V_{eff})_{min}$ only $r = r_0$ is possible therefore orbit is circular.

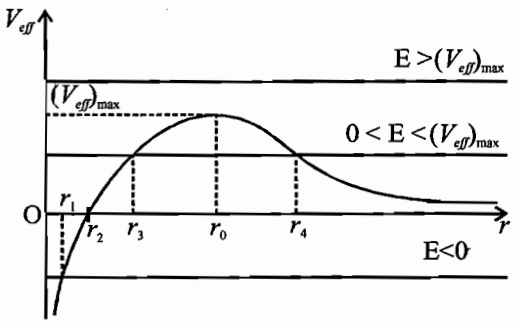
(ii) For $E > (V_{eff})_{min}$, $r_{min} = r_1, r_{max} = r_2$ that is $r_1 \le r \le r_2$. Therefore orbit is bounded. Thus we see that under Hooke's force orbit is bounded for all positive energies which are greater than the minimum value of effective potential.

(iii) $E < (V_{eff})_{min}$ is not allowed

Dynamics under $F = \frac{-k}{r^4}$ force or $V(r) = \frac{-k}{3r^3}$:

$$V_{eff}(r) = \frac{-k}{3r^2} + \frac{L^2}{2mr^2}$$

Plot of $V_{eff}(r)$ versus r is shown in the figure.



CAREER ENDEAVOUR

From the plot we draw following conclusion.

(i) For E < 0, $r_{min} = 0$, $r_{max} = r_1$ that is $0 \le r \le r_1$. Therefore motion is bounded.

(ii) For E = 0, $0 \le r \le r_2$. Therefore motion is bounded.

(iii) For $0 < E < (V_{eff})_{max}$, $0 \le r \le r_3$ or $r_4 \le r \le \infty$. Therefore for such value of energy, motion may be bounded or unbounded.

(iv) For $E \ge (V_{eff})_{max}$, $0 \le r \le \infty$. Therefore motion is unbounded.

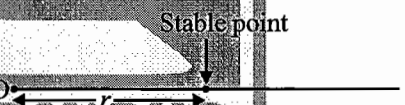
(v) This type of potential has a special feature.

- For $r > r_0$ $\frac{\partial V_{eff}}{\partial r} < 0$. Therefore $f_{eff} = \frac{-\partial V_{eff}}{\partial r} > 0$ i.e. force on the particle is away from the centre of force.
- For $r < r_0$ $\frac{\partial V_{eff}}{\partial r} > 0$. Therefore $f_{eff} = \frac{-\partial V_{eff}}{\partial r} < 0$ i.e. force on the particle is towards the centre of force.
- Due to attractive nature of force for $r < r_0$ any object coming closer to the centre of force gets captured if it is able to cross the $r = r_0$ point.

Example: A planet of mass m and angular momentum L moves in a circular orbit in a potential, $V(r) = -k/r$ where k is a positive constant find the radius of stable circular orbit. If the planet is slightly perturbed, find angular frequency of radial oscillation.

Soln. $V_{eff} = \frac{L^2}{2mr^2} + V(r) = \frac{L^2}{2mr^2} - \frac{k}{r}$

For stable point (orbit) $\left.\frac{\partial V_{eff}}{\partial r}\right|_{r=r_0} = 0$

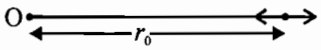


$$\therefore -\frac{L^2}{mr_0^3} + \frac{k}{r_0^2} = 0 \text{ or } r_0 = \frac{L^2}{mk}$$

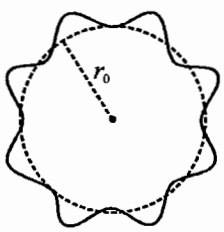
Therefore radius of stable circular orbit is L^2/mk

Angular frequency of oscillation about stable point:

$$\omega = \sqrt{\frac{\left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r=r_0}}{m}} = \sqrt{\frac{\left(\frac{3L^2}{mr_0^4} - \frac{2k}{r_0^3}\right)}{m}}$$

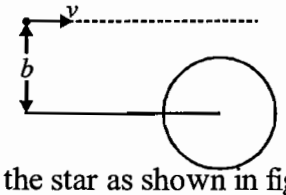


$$= \sqrt{\frac{\frac{1}{r_0^3}\left(\frac{3L^2}{m.\frac{L^2}{mk}} - 2k\right)}{m}} = \sqrt{\frac{k}{mr_0^3}} = \sqrt{\frac{k^4 m^2}{L^6}} = \frac{mk^2}{L^3}$$



actual radial oscillation is as shown in figure.

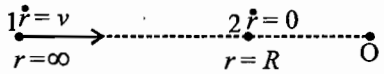
Example: A small asteroid is approaching a massive star with a speed v from very large distance, at an impact parameter 'b' as shown in figure. If the mass of the star is M and its radius is R, then what is the minimum value of b such that the asteroid will miss the star?

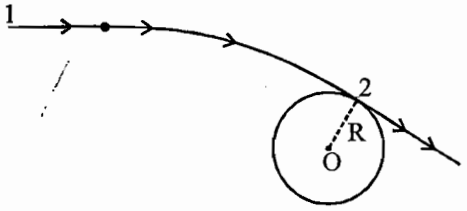


Soln. Angular momentum of asteroid about centre of star is $L = mvb$

Effective potential energy $V_{eff} = \frac{-GMm}{r} + \frac{L^2}{2mr^2} = \frac{-GMm}{r} + \frac{mv^2b^2}{2r^2}$

For minimum value of b we will have to assume that asteroid just misses the star as shown in figure. Equivalent 1-d problem is also shown in figure





Total energy is conserved in central force motion

Therefore, $E_1 = E_2$

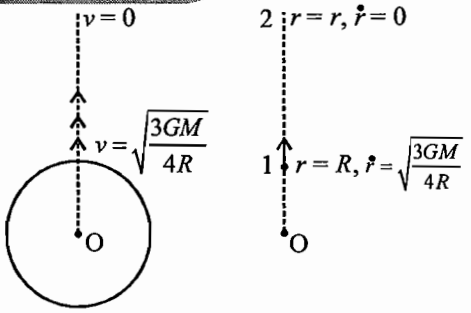
or $\left(\frac{1}{2}m\dot{r}^2 + V_{eff}\right)_{\text{at } r=\infty} = \left(\frac{1}{2}m\dot{r}^2 + V_{eff}\right)_{\text{at } r=R}$

$$\frac{1}{2}mv^2 - \frac{GMm}{\infty} + \frac{mv^2b^2}{\infty} = \left(\frac{1}{2}mo^2 - \frac{GMm}{R} + \frac{mv^2b^2}{2R^2}\right)$$

$$\therefore \quad v^2 + \frac{2GM}{R} = \frac{v^2b^2}{R^2} \quad \text{or} \quad b = R\sqrt{1 + \frac{2GM}{Rv^2}}$$

Example: A particle is thrown in radially outward direction from earth's surface with initial speed $\sqrt{\frac{3GM}{4R}}$ where M is mass of earth R is radius of earth. Find the height upto which particle goes.

Soln. Particle has been thrown radially outward



therefore $L = 0$, $V_{eff} = V(r) + \frac{L^2}{2mr^2}$

$$\therefore \quad V_{eff} = \frac{-GMm}{r} + 0$$

from conservation of energy we get $E_1 = E_2$

or $\left(\frac{1}{2}m\dot{r}^2 + V_{eff}\right)_{r=R} = \left(\frac{1}{2}m\dot{r}^2 + V_{eff}\right)_{r=r}$

$$\frac{1}{2}m.\frac{3GM}{4R} - \frac{GMm}{R} = 0 - \frac{GMm}{r}$$

$$\frac{3}{8R} - \frac{1}{R} = -\frac{1}{r} \quad \therefore \; r = \frac{8R}{5}$$

Therefore, maximum height from earth's surface = $r - R = \frac{3R}{5}$

CAREER ENDEAVOUR

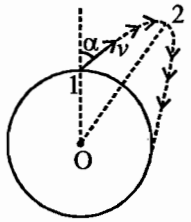
Example: A particle is thrown from the earth's surface with speed $\sqrt{\frac{GM}{R}}$ where M = mass of earth and R = radius of earth. If direction of initial velocity makes an angle α with the outward radial direction, find the maximum distance of the particle from centre of earth.

Soln. Angular momentum of the particle is

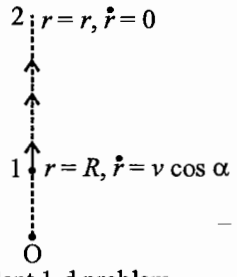
$$L = mv\sin\alpha R = m\sqrt{\frac{GM}{R}}\sin\alpha\, R$$

$$= m\sqrt{GMR}\sin\alpha$$

$$\therefore\ V_{eff} = V(r) + \frac{L^2}{2mr^2} = \frac{-GMm}{r} + \frac{GMmR}{2r^2}\sin^2\alpha$$



actual



equivalent 1-d problem

From conservation of energy we get $E_1 = E_2$

or $\left(\frac{1}{2}m\dot{r}^2 + V_{eff}\right)_{r=R} = \left(\frac{1}{2}m\dot{r}^2 + V_{eff}\right)_{r=r}$ (see the figure)

$$\frac{1}{2}m\frac{GM}{R}\cos^2\alpha - \frac{GMm}{R} + \frac{GMm}{2R}\sin^2\alpha = \frac{-GMm}{r} + \frac{GMmR}{2r^2}\sin^2\alpha$$

Divide both sides by $\frac{GMm}{2R}$ to get $\cos^2\alpha - 2 + \sin^2\alpha = -2\frac{R}{r} + \frac{R^2}{r^2}\sin^2\alpha$

or $-1 = -2\left(\frac{R}{r}\right) + \left(\frac{R}{r}\right)^2\sin^2\alpha$ $\quad\therefore\ \left(\frac{R}{r}\right)^2\sin^2\alpha - 2\left(\frac{R}{r}\right) + 1 = 0$

$$\therefore\ \frac{R}{r} = \frac{+2 \pm \sqrt{4 - 4\sin^2\alpha}}{2\sin^2\alpha} \qquad \frac{R}{r} = \frac{+1 \pm \cos\alpha}{\sin^2\alpha}$$

$$\therefore\ r = \frac{R\sin^2\alpha}{1 \pm \cos\alpha} = \frac{R(1-\cos^2\alpha)}{(1 \pm \cos\alpha)} \qquad \therefore\ r = R(1-\cos\alpha) \quad \text{or} \quad r = R(1+\cos\alpha)$$

CAREER ENDEAVOUR

Since r must be greater than r. Therefore, $r = R(1+\cos\alpha)$

Condition for stable orbit: If V_{eff} versus r plot has a minimum then stable circular orbit can exist.

Let the central force is $f(r) = -kr^n$

Therefore, $V_{eff} = V(r) + \frac{L^2}{2mr^2} = -\int f(r)dr + \frac{L^2}{2mr^2} = \frac{kr^{n+1}}{n+1} + \frac{L^2}{2mr^2}$

Condition for existence of minimum is

$$\left.\frac{\partial V_{eff}}{\partial r}\right|_{r=r_0} = 0 \qquad \text{... (i)}$$

and $$\left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r=r_0} > 0 \qquad \text{... (ii)}$$

using first condition we get $+kr_0^n - \frac{L^2}{mr_0^3} = 0$ or $r_0^{n+3} = \frac{L^2}{mk}$ ... (iii)

using second condition we get, $nkr_0^{n-1} + \frac{3L^2}{mr_0^4} > 0$ or $\frac{k}{r_0^4}\left(nr_0^{n+3} + \frac{3L^2}{mk}\right) > 0$

using equation (iii) we get, $\frac{k}{r_0^4}\left(nr_0^{n+3} + 3r_0^{n+3}\right) > 0$ or $kr_0^{n-1}(n+3) > 0$

Since k and r_0 are positive therefore we must have $n+3>0$ or $\boxed{n>-3}$

Thus, if $F = -kr^n$ then n must be greater than -3 for orbit to be stable.

Condition for closed orbit (Bertrand's theorem): If $F(r) = -kr^n$ then closed orbits are possible only for $n=1$ and $n=-2$ i.e. for $f=-kr$, Hooke's force and for $f(r) = \frac{-k}{r^2}$, inverse square force (gravitation).

Virial theorem: If $f \propto r^n$ then average value of potential energy and kinetic energy are related as

$$< K.E > = \frac{n+1}{2} < P.E >$$

for gravitational force $n=-2$ $\therefore$ $< K.E > = -\frac{1}{2} < P.E >$

for Hooke's force $n=1$ $\therefore$ $< K.E > = < P.E >$

Example: A particle moves under a central potential $V(r) = \frac{-k}{r^m}$. What should be value of m for its orbit to be stable.

Soln. Corresponding is $f(r) = -\frac{\partial V}{\partial r} = \frac{-mk}{r^{m+1}} = -mkr^{-(m+1)}$

We know that for $f = -kr^n$ condition for stability is $n > -3$.

Therefore for orbit to be stable under given potential we must have,

$-(m+1) > -3$ or $m+1 < 3$ $\therefore m < 2$

Example: A particle is moving under a central force $f(r) = \frac{-k}{r^{3-n}}$ for what values of n orbit of the particle can be closed.

Soln. We know that for $f(r) = -kr^m$ orbit is closed if $m=-2$ or 1.

Therefore, $-(3-n) = -2 \Rightarrow n = 1$ and $-(3-n) = 1 \Rightarrow n = 4$.

Therefore, values of n for which orbit can be closed are 1 and 4.

Keppler's law of planetary motion: Keppler had given three empirical laws of planetary motion on the basis of analysis of astronomical data collected by his master Tycho Brahe.

First Law: All the planets move in elliptical orbit around the sun with sun being at one focus of the ellipse.

Second Law: Line joining the centre of planet to the centre of the sun sweeps equal area in equal interval of time. That is areal velocity of a planet is constant.

Third Law: Square of time of revolution is proportional to cube of semi-major axis of elliptical orbit.

i.e. $T^2 \propto a^3$

CAREER ENDEAVOUR

The Laplace-Runge-Lenz Vector:

It is conserved in Kepler's Problem.

For central force, the Newton's law is

$$\dot{\vec{p}} = f(r)\frac{\vec{r}}{r}$$

Now, consider

$$\frac{d(\vec{p}\times\vec{L})}{dt} = \vec{p}\times\dot{\vec{L}} + \dot{\vec{p}}\times\vec{L} = \dot{\vec{p}}\times\vec{L}$$

Because, $\vec{L}$ is constant, so $\frac{d\vec{L}}{dt} = 0$

So, we get

$$\frac{d(\vec{p}\times\vec{L})}{dt} = \dot{\vec{p}}\times\vec{L} = \frac{mf(r)}{r}\left[\vec{r}\times(\vec{r}\times\dot{\vec{r}})\right] \qquad \text{(because } \vec{L} = m\vec{r}\times\dot{\vec{r}}\text{)}$$

The above quation can be further simplified by noting that

$$\vec{r}\cdot\dot{\vec{r}} = \frac{1}{2}\frac{d}{dt}(\vec{r}\cdot\vec{r}) = r\dot{r}$$

So
$$\frac{d}{dt}(\vec{p}\times\vec{L}) = -mf(r)r^2\left(\frac{\vec{r}}{r} - \frac{\vec{r}\dot{r}}{r^2}\right)$$

Or,
$$\frac{d}{dt}(\vec{p}\times\vec{L}) = -mf(r)r^2\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$

Now, for inverse square law of force, i.e. $f(r) = -\frac{k}{r^2}$, so

$$\frac{d}{dt}(\vec{p}\times\vec{L}) = \frac{d}{dt}\left(\frac{mk\vec{r}}{r}\right)$$

We see that,

$$\frac{d}{dt}\left(\vec{p}\times\vec{L} - \frac{mk\vec{r}}{r}\right) = 0$$

which says that for the Kepler problem there exists a conserved vector $\vec{A}$ defined as

$$\vec{A} = \vec{p}\times\vec{L} - mk\frac{\vec{r}}{r}$$

The conserved quantity $\vec{A}$ is called Laplace-Runge-Lenz vector. We see that $\vec{A}$ lies in the plane of the borbit, because $\vec{L}$ is perpendicular to plane of orbit, so, $\vec{p}\times\vec{L}$ is perpendicular to $\vec{L}$ (i.e. in plane of orbit). Also,

$$\vec{A}\cdot\vec{L} = 0$$

If θ is used to denote the angle between $\vec{r}$ and fixed direction of $\vec{A}$, then

$$\vec{A}\cdot\vec{r} = Ar\cos\theta = \vec{r}\cdot\left(\vec{p}\times\vec{L} - mk\frac{\vec{r}}{r}\right)$$

Or
$$\vec{A}\cdot\vec{r} = Ar\cos\theta = \vec{r}\cdot(\vec{p}\times\vec{L}) - mkr$$

CAREER ENDEAVOUR

Now, by permutation of the terms in the triple dot product, we have

$$\vec{\mathbf{r}}\cdot\left(\vec{\mathbf{p}}\times\vec{\mathbf{L}}\right)=\vec{\mathbf{L}}\cdot\left(\vec{\mathbf{r}}\times\vec{\mathbf{p}}\right)=\ell^2 \qquad \left[\because\ \vec{\mathbf{a}}\cdot\left(\vec{\mathbf{b}}\times\vec{\mathbf{c}}\right)=\vec{\mathbf{b}}\cdot\left(\vec{\mathbf{c}}\times\vec{\mathbf{a}}\right)=\vec{\mathbf{c}}\cdot\left(\vec{\mathbf{a}}\times\vec{\mathbf{b}}\right)\right]$$

So, we have

$$Ar\cos\theta=\ell^2-mkr$$

Or, $$\frac{1}{r}=\frac{mk}{\ell^2}\left(1+\frac{A}{mk}\cos\theta\right)$$

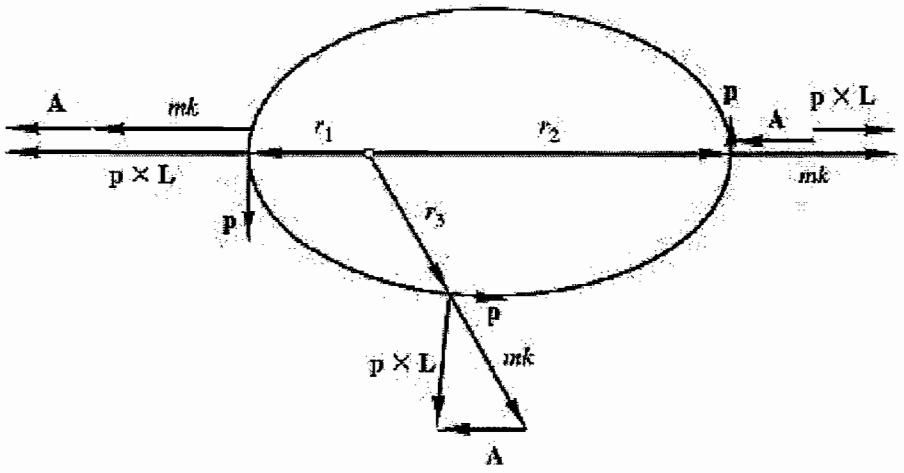
So, $\vec{\mathbf{A}}$ gives another method to derive the equation of orbit. Comparing this with equation of orbit derived earlier, $A=mke$

Since $$e=\sqrt{1+\frac{2E\ell^2}{mk^2}}$$

We have, $$A=mk\sqrt{1+\frac{2E\ell^2}{mk^2}}$$

Or, $$A^2=m^2k^2\left(1+\frac{2E\ell^2}{mk^2}\right)$$

Or $$A^2=m^2k^2+2mE\ell^2$$



Rigid body dynamics

A rigid body is collection of large number of particles. Distance between each pair of particle remains constant. i.e. $|\vec{r}_i - \vec{r}_j|$ = constant. Rotational dynamics of rigid body involves basically dynamics of large number of particles. Therefore it is not possible to describe the dynamics in terms of movement of each particle. However, by introducing concept of moment of inertia or inertia tensor rotational dynamics of a rigid body can be systematically discussed. Moment of inertia or inertia tensor collectively takes into account the movement of each particle.

Moment of inertia: Moment of inertia of a system of particles about an axis of rotation is defined as

$I = \sum_i m_i r_i^2$ (for discrete case)

where m_i = mass of each point particle

r_i = perpendicular distance from the axis of rotation

or $I = \int r_\perp^2 dm$ (for continuous case)

where dm = elementary mass

$r_\perp$ = perpendicular distance of elementary mass from the axis of rotation.

Example: Two particles of masses m_1 and m_2 are kept 'r' distance apart. Calculate moment of inertial of the system about an axis passing through the centre of mass and perpendicular to the line joining the masses.

Soln. Let $m_1 > m_2$, therefore centre of mass of the system lies nearer to m_1.

$\therefore \quad I = \sum m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2$

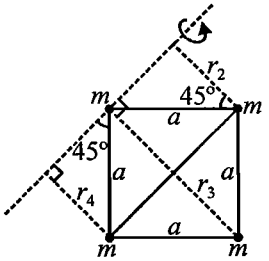
Since, $r_1 = \dfrac{m_2 r}{m_1 + m_2}$ and $r_2 = \dfrac{m_1 r}{m_1 + m_2}$

$\therefore \quad I = m.\left(\dfrac{m_2 r}{m_1 + m_2}\right)^2 + m_2\left(\dfrac{mr}{m_1 + m_2}\right)^2 = \dfrac{m_1 . m_2}{m_1 + m_2} r^2$

CAREER ENDEAVOUR

Example: Four particles each of mass 'm' are kept at the corners of a square of side 'a'. Calculate moment of inertia about an axis passing through one corner. The axis is parallel to diagonal of square.

Soln. $I = \sum_i m_i r_i^2 = m(r_1^2 + r_2^2 + r_3^2 + r_4^2)$



from figure shown, $r_1 = 0, r_2 = a\cos 45°, r_3 = a\sqrt{2}, r_4 = a\sin 45°$

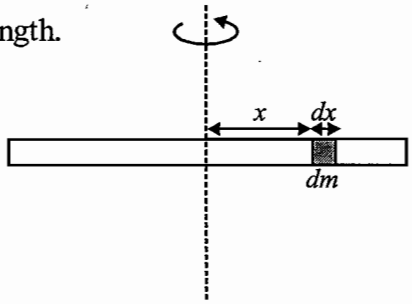
$\therefore \quad I = m\left[0 + \left(\dfrac{a}{\sqrt{2}}\right)^2 + (a\sqrt{2})^2 + \left(\dfrac{a}{\sqrt{2}}\right)^2\right] = 3ma^2$

Moment of inertia of a uniform thin rod: Let M = mass of rod and L = length of the rod.

Moment of inertia of a continuous mass distribution is defined as $I = \int r_\perp^2 dm$

Let m take an element of width dx at a distance x from the axis of rotation. Therefore $dm = \frac{M}{L}dx$

Case 1: About an axis passing through its centre and $\perp$ to its length.



Here, $r_\perp = x$

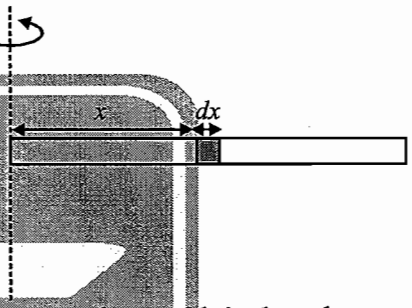
$$\therefore \quad I = \int_{-L/2}^{+L/2} x^2 \frac{M}{L} dx = \frac{M}{L}\int_{-L/2}^{L/2} x^2 dx$$

$$= \frac{ML^2}{12} \quad \therefore \boxed{I = \frac{ML^2}{12}}$$

Case 2: About an axis through its end and perpendicular to its length

Here, $r_\perp = x$

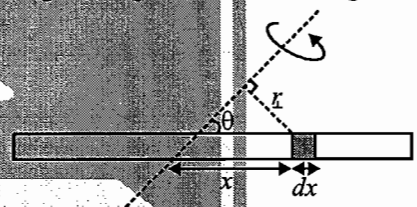
$$\therefore \quad I = \int_0^L x^2 \frac{M}{L} dx$$



$$= \frac{ML^2}{3} \quad \therefore \boxed{I = \frac{ML^2}{3}}$$

Case 3: About an axis passing through its centre and making an angle θ with its length.

Here, $r_\perp = x\sin\theta$



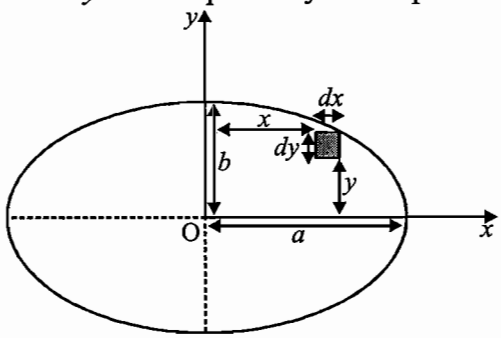
$$\therefore \quad I = \int_{-L/2}^{L/2} x^2 \sin^2\theta . \frac{M}{L} dx = \frac{M}{L}\sin^2\theta \int_{-L/2}^{L/2} x^2 dx$$

$$= \frac{ML^2}{12}\sin^2\theta \quad \therefore \boxed{I = \frac{ML^2}{12}\sin^2\theta}$$

CAREER ENDEAVOUR

Moment of inertia of uniform elliptical thin plate: Let M = mass of plate and a, b be the semi-major and minor axes respectively.

If centre of ellipse is take as origin and major and minor axes as x and y axes respectively then equation of ellipse is given as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Let us take an element at point (x, y). Let its area be $dx.dy$

therefore, $dm = \frac{M}{A}.dx.dy = \frac{M}{\pi ab}dx.dy$

where $A = \pi ab$ = area of ellipse.

Case 1: About the major axis

Here, $r_\perp = y$

$$\therefore \quad I = \int r_\perp^2 dm = \iint y^2 . \frac{M}{\pi ab} dxdy = \frac{M}{\pi ab}\iint y^2 dx.dy = \frac{M}{\pi ab}\int dx \int y^2 dy$$

From equation of ellipse $y = \pm b\sqrt{1-\frac{x^2}{a^2}}$

$$\therefore I = \frac{M}{\pi ab}\int_{-a}^{a} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} y^2 dy = \frac{2M}{3\pi ab}\int_{-a}^{a} dx\left(b\sqrt{1-\frac{x^2}{a^2}}\right)^3 = \frac{2Mb^2}{3\pi a}\int_{-a}^{a}\left(1-\frac{x^2}{a^2}\right)^{3/2} dx = \frac{4Mb^2}{3\pi a}\int_{0}^{a}\left(1-\frac{x^2}{a^2}\right)^{3/2} dx$$

The value of integral is $\frac{3\pi}{16}a$

Therefore, $I = \frac{4Mb^2}{3\pi a}.\frac{3\pi}{16}a = \frac{Mb^2}{4}$ or $\boxed{I = \frac{Mb^2}{4}}$ about major axis

Case 2: About the minor axis

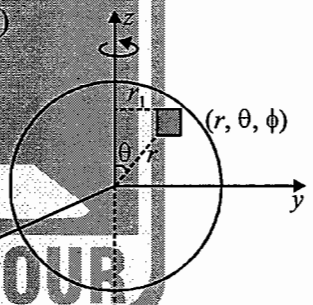
Here $r_\perp = x \quad \therefore \; I = \int r_\perp^2 d_m = \iint x^2 \frac{M}{\pi ab} dx\, dy$

A similar calculation gives $\boxed{I = \frac{Ma^2}{4}}$ about minor axis

Note: It is easier to use plane polar, spherical polar and cylindrical coordinates for objects having circular symmetry, spherical symmetry and cylindrical symmetry. Therefore depending on the symmetry we will use convenient coordinate system.

Moment of inertia of a uniform solid sphere about its diameter: Let M = mass of the sphere and R = radius of sphere. Let us take an element at a point (r, θ, ϕ)

Volume of element $(dv) = r^2 \sin\theta d\theta d\phi dr$



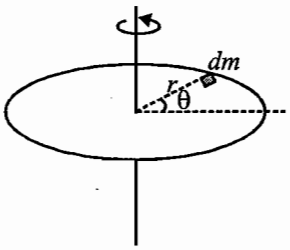
Therefore, $dm = \frac{M}{V}dV = \frac{M}{\frac{4}{3}\pi R^3}.r^2 dr \sin\theta d\theta d\phi$

Here $r_\perp = r\sin\theta$ (as shown in figure)

$$\therefore \quad I = \int r_\perp^2 + dm = \iiint r^2 \sin^2\theta . \frac{M}{\frac{4}{3}\pi R^3} r^2 dr \sin\theta d\theta d\phi \quad = \frac{3M}{4\pi R^3}\int_0^R r^4 dr \int_0^\pi \sin^3\theta d\theta \int_0^{2\pi} d\phi$$

$$= \frac{3M}{4\pi R^3}.\frac{R^5}{5}.\frac{4}{3}.2\pi = \frac{2}{5}MR^2 \qquad \therefore \; \boxed{I = \frac{2}{5}MR^2}$$

Moment of inertia of uniform circular disc: Let M = mass of disc, R = radius of the disc, let us take an element at (r, θ). Area of element $(dA) = rdrd\theta$



Mass of element $(dm) = \frac{M}{A}.dA = \frac{M}{\pi R^2} rdrd\theta$

here $r_\perp = r$

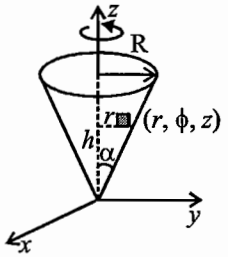
Therefore, $I = \int r_\perp{}^2 dm = \iint r^2 . \frac{M}{\pi R^2} rdrd\theta = \frac{M}{\pi R^2}\int_0^R r^3 dr \int_0^{2\pi} d\theta$

CAREER ENDEAVOUR

$$= \frac{M}{\pi R^2} \cdot \frac{R^4}{4} .2\pi \qquad \therefore \boxed{I = \frac{MR^2}{2}}$$

Moment of inertia of uniform right circular cone about its axis: Let M = mass, R = base radius, h = height of the cone. Let us take an element at (r, ϕ, z) in cylindrical coordinate system.

Volume of element $(dV) = rdrd\phi dz$



Mass of the element $(dm) = \dfrac{M}{\frac{1}{3}\pi R^2 h} .rdrdzd\phi$

Here $r_\perp = r$

Therefore, $I = \int r_\perp^2 dm = \iiint r^2 . \dfrac{M}{\frac{1}{3}\pi R^2 h} rdrdz\phi = \dfrac{3M}{\pi R^2 h} \int dz \int r^3 dr \int_0^{2\pi} d\phi$

At the slanting surface of cylinder $\tan\alpha = \dfrac{r}{z}$ or $r = z\tan\alpha$

$$\therefore \; I = \frac{3M}{\pi R^2 h} \int_0^h dz \int_0^{z\tan\alpha} r^3 dr \int_0^{2\pi} d\phi = \frac{3M}{\pi R^2 h} . \int_0^h dz . \frac{z^4 \tan^4\alpha}{4} .2\pi$$

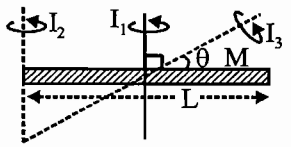
$$= \frac{3M}{2R^2 h} \tan^4\alpha \int_0^h z^4 dz = \frac{3M}{2R^2 h} . \frac{R^4}{h^4} . \frac{h^5}{5} = \frac{3}{10} MR^2 \qquad \therefore \quad \boxed{I = \frac{3}{10} MR^2}$$

CAREER ENDEAVOUR

List of moment of inertia of some objects:

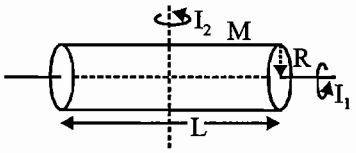
Object	Moment of inertia
Thin circular ring	$I_1 = MR^2, I_2 = \dfrac{MR^2}{2}$
Thin circular disc	$I_1 = \dfrac{MR^2}{2}, I_2 = \dfrac{MR^2}{4}$
Annular ring	$I = \dfrac{1}{2} M(R_1^2 + R_2^2)$

Thin rod



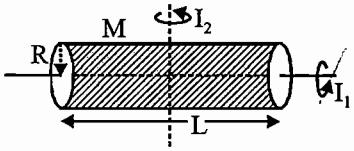
$$I_1 \frac{ML^2}{12}, I_2 = \frac{ML^2}{3}, I_3 = \frac{ML^2}{12}\sin^2\theta$$

Thin hollow cylinder



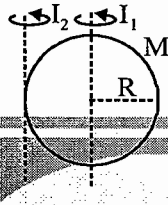
$$I_1 = MR^2, I_2 = M\left(\frac{R^2}{2} + \frac{L^2}{12}\right)$$

Solid cylinder



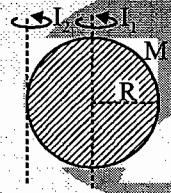
$$I_1 = \frac{MR^2}{2}, I_2 = M\left(\frac{R^2}{4} + \frac{L^2}{12}\right)$$

Thin hollow sphere



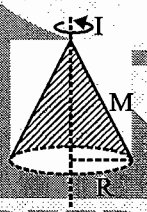
$$I_1 = \frac{2}{3}MR^2, I_2 = \frac{5}{3}MR^2$$

Solid sphere



$$I_1 = \frac{2}{5}MR^2, I_2 = \frac{7}{5}MR^2$$

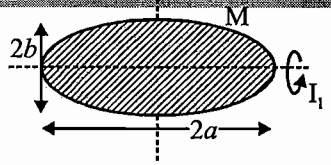
Right circular solid cone



$$I = \frac{3}{10}MR^2$$

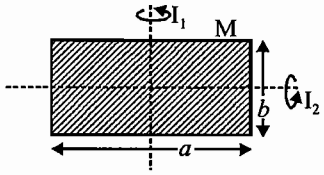
CAREER ENDEAVOUR

Elliptical plate



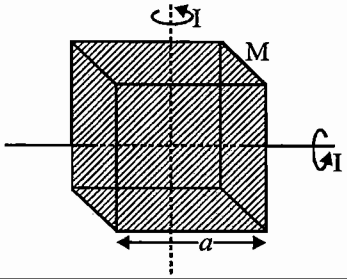
$$I_1 = \frac{Mb^2}{4}, I_2 = \frac{Ma^2}{4}$$

Rectangular plate

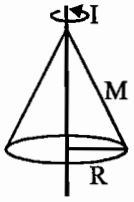


$$I_1 = \frac{Ma^2}{12}, I_2 = \frac{Mb^2}{12}$$

Solid cube



$$I = \frac{Ma^2}{6}$$

Right circular hollow cone 

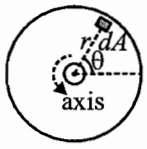
$$I = \frac{MR^2}{2}$$

Example: A thin circular disc of mass 'M' and radius 'R' has non-uniform mass density which varies linearly with distance from centre. Calculate moment of inertia of the disc about its axis.

Soln. Let us taken an element at (r, θ)

Area of element $(dA) = rdrd\theta$

Mass of element $(dm) = \sigma dA$



where σ = surface mass density. According to question $\sigma \alpha r$.

Therefore, $\sigma = kr$, k = constant

$\therefore \quad dm = \sigma dA = kr^2 drd\theta$

$$\therefore \quad M = \iint kr^2 drd\theta = k\int_0^R r^2 dr \int_0^{2\pi} d\theta = \frac{2\pi kR^3}{3} \quad \text{or} \quad k = \frac{3M}{2\pi R^3} \qquad \text{... (a)}$$

Moment of inertia about the axis of disc (axis passing through centre and perpendicular to its plane)

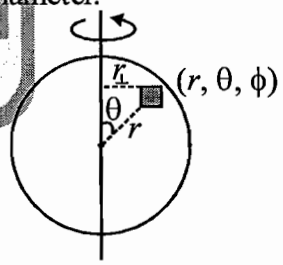
$$I = \int r_\perp^2 dm = \iint r^2 kr^2 drd\theta = k\int_0^R r^4 dr \int_0^{2\pi} d\theta = 2\pi k \frac{R^5}{5}$$

Using value of k from (a) we get, $I = \frac{3}{5}MR^2$

Example: A solid sphere of mass M and radius R has non-uniform mass density which varies linearly with distance from centre. Calculate moment of inertia of the sphere about its diameter.

Soln. According to question, $\rho \alpha r$ or $\rho = kr$

Let us take an element at (r, θ, ϕ)



Mass of the element $dm = \rho dv = kr.r^2 dr \sin\theta d\theta d\phi$

$$\therefore M = \iiint kr^3 dr \sin\theta d\theta d\phi = k\int_0^R r^3 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = \pi kR^4$$

$$\therefore \quad k = \frac{M}{\pi R^4} \qquad \text{... (a)}$$

Moment of inertia about its diameter

Here $r_\perp = r\sin\theta$

$$I = \int r_\perp^2 dm = \iiint r^2 \sin^2\theta . kr^3 dr \sin\theta . d\theta d\phi$$

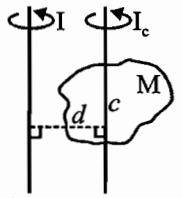
$$= k\int_0^R r^5 dr \int_0^\pi \sin^3\theta d\theta . \int_0^{2\pi} d\phi = k\frac{R^6}{6}.\frac{4}{3}.2\pi = \frac{4\pi kR^6}{9}$$

using value of k from (a) we get, $I = \frac{4MR^2}{9}$

CAREER ENDEAVOUR

Parallel axes theorem: It states that if I_c be moment of inertia of an object about an axis passing through its centre of mass then moment of inertia about any other parallel axis which is at a separation 'd' is given as

$$I = I_c + Md^2$$

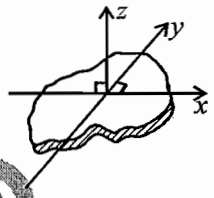


where M = mass of object

This theorem is valid for all types of objects ($1d$, $2d$, $3d$)

Perpendicular axes theorem: This theorem is applicable for $1d$ or $2d$ objects. It states that if x and y axes lie in the plane of object and z axis is perpendicular to its plane then

$$I_{zz} = I_{xx} + I_{yy}$$



The three axes must intersect at common point, however there is no condition as such that the axes should pass through centre of mass.

Therefore the axes may or may not pass through centre of mass.

Example: A uniform solid cube has mass M and side 'a'. Calculate moment of inertia of the cube about an axis which coincides with one of its sides.

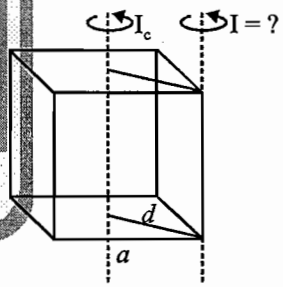
Soln. The axis given in question and a parallel axis through centre of mass are shown in the figure.

separation between two axes is $d = \frac{a}{\sqrt{2}}$

we know that moment of inertia of cube about an axis through its centre and perpendicular to its face is $I_c = \frac{Ma^2}{6}$

Therefore using parallel axes theorem we get

$$I = I_c + Md^2 = \frac{Ma^2}{6} + \frac{Ma^2}{2} = \frac{2}{3}Ma^2$$

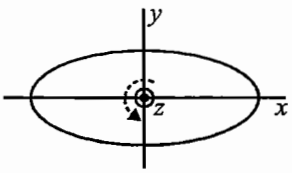


CAREER ENDEAVOUR

Example: An elliptical plate has mass M and semi-major and minor axes 'a' and 'b' respectively. Calculate moment of inertia of the plate about an axis perpendicular to its plane and passing through its centre.

Soln. Let us take semi-major and minor axes as x and y axes and the axis given in the question as z axis.

$$\therefore \quad I_{xx} = \frac{Mb^2}{4}, I_{yy} = \frac{Ma^2}{4}$$



From perpendicular axes theorem

$$I_{zz} = I_{xx} + I_{yy} = \frac{M}{4}(a^2 + b^2)$$

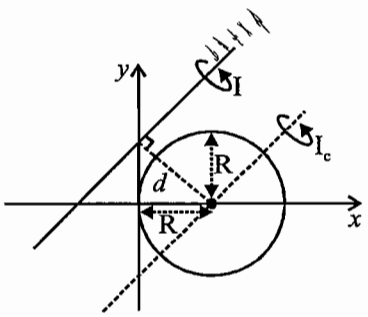
Example: A thin circular ring of mass M and radius R lies in x-y plane. Equation of the ring is $(x-R)^2 + y^2 = R^2$. Calculate moment of inertia of the ring about a straight line $y = x + R$.

Soln. The ring and the straight line are shown in the figure. To calculate moment of inertia about the given axis we draw a parallel line through centre of ring and apply parallel axes theorem.

$\therefore I = I_c + Md^2$

$= \frac{MR^2}{2} + M(R\sqrt{2})^2$

$= MR^2 + 2MR^2 = 3MR^2$



Example: Moment of inertia of a thin circular ring of unit mass lying in x-y plane is 2 unit and 3 unit respectively about the straight lines $y = x + 3$ and $y = -x + 2$. If centre of the ring lies at (1, 1) calculate moment of inertia of the ring about the axis perpendicular to its plane and passing through its centre.

Soln. Point of intersection of the two straight lines is $P\left(-\frac{1}{2}, \frac{5}{2}\right)$. Distance of point of intersection from centre of the circle is $d = \sqrt{\left(1+\frac{1}{2}\right)^2 + \left(1-\frac{5}{2}\right)^2} = \frac{3}{\sqrt{2}}$. Moment of inertia of the ring about an axis passing through point P and perpendicular to x-y plane is (use perpendicular axes theorem)

I = 2 + 3 = 5

The perpendicular axis through P is parallel to perpendicular axis through the centre of the ring.

Therefore using parallel axis theorem we get

$I = I_c + Md^2$

$\therefore \quad 5 = I_c + 1 \times \frac{9}{2}$ or $I_c = 0.5$

Dynamics of rigid body: A rigid body can have following types of dynamics

(i) Pure translation: If all the points of an object undergo equal displacement in equal interval of time then dynamics is called pure translation. Pure translational motion of a rigid body is equivalent to translation of centre of mass of the object. Therefore,

Kinetic energy $= \frac{1}{2}MV_c^2$, momentum $= MV_c$

Equation of motion $\vec{F} = M\vec{a}_c$

where v_c and a_c are velocity and acceleration of centre of mass.

(ii) Pure rotation: In this type of motion at least one point of rigid body remains stationary. Rotational motion of a rigid body cannot be described in terms of its outer of mass. Instead dynamics of each particle has to be considered. That is why concept of moment of inertia comes into play in rotational motion.

In pure rotation distance of each particle of rigid body remains constant from the point or axis about which rotation takes place.

In pure rotational motion we use following terms for a rigid body.

Kinetic energy $= \frac{1}{2}I\omega^2$, where I = moment of inertia about given axis

Angular momentum about the axis $L = I\omega$

Equation of motion $\tau = I\alpha$ (about the axis)

where $\alpha = \frac{d\omega}{dt}$, α = angular acceleration, ω = angular velocity

and $\omega = \frac{d\theta}{dt}$, θ = angular displacement

If α = constant then $\omega = \omega_0 + \alpha t,\ \omega^2 = \omega_0^2 + 2\alpha\theta,\ \theta = \omega_0 t + \frac{1}{2}\alpha t^2$

where ω_0 = initial angular velocity, ω = final angular velocity

(iii) General motion: In this case object undergoes translation as well rotation. There is no point which remains stationary for all the time. And different points of the object undergo different displacement. According to Chasle's theorem a general motion is equivalent to translation of centre of mass plus pure rotation about centre of mass i.e.

General motion = Pure translation of centre of mass + Pure rotation about centre of mass

Therefore in general motion we use following terms for a rigid body.

Kinetic energy = $\frac{1}{2}MV_c^2 + \frac{1}{2}I_c\omega^2$

Equation of motion $\vec{F} = Ma_c,\ \tau = I_c\alpha_c$

Angular momentum $\vec{L} = I_c\vec{\omega} + M\vec{r} \times \vec{v}_c$. If the rigid body is round in shape and it is moving on a surface then about any point on the path of the object angular momentum is written as $L = I_c\omega + Mv_cR$ where R is radius of object.

(iv) Pure rolling: It is a special kind of general motion of round objects in which following relations are satisfied.

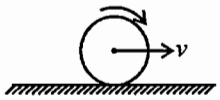
$\left.\begin{aligned} v_c &= \omega R \\ a_c &= \alpha R \end{aligned}\right\}$ condition for rolling

In most cases friction is necessary for rolling motion. However there is no loss of energy against friction in rolling motion. Therefore total mechanical energy remains conserved in rolling motion all though friction is present.

CAREER ENDEAVOUR

Example: A hollow sphere of mass M and radius R rolls on a horizontal surface such that centre of the sphere moves with speed v. Calculate kinetic energy and angular momentum of the sphere.

Soln. Given, $v_c = v$ and sphere is rolling



Therefore we use condition of rolling ($v_c = \omega R$)

To find its angular velocity

$$\therefore \omega = \frac{v_c}{R} = \frac{v}{R}$$

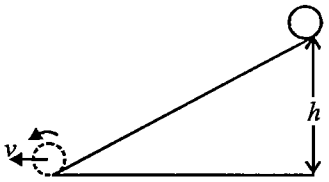
$$\text{Kinetic energy} = \frac{1}{2}Mv_c^2 + \frac{1}{2}I_c\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}.\frac{2}{3}MR^2.\frac{v^2}{R^2} = \frac{5}{6}Mv^2$$

$$\text{Angular momentum} = I_c\omega + Mv_cR = \frac{2}{3}MR^2.\frac{v}{R} + MvR = \frac{5}{3}MvR$$

Example: A hollow sphere rolls down an inclined plane from a height '*h*' what is speed of the sphere when it reaches the bottom of inclined plane.

Soln. There must be friction on the plane. However total mechanical energy is conserved in rolling motion. Therefore, loss in potential energy = gain in kinetic energy

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

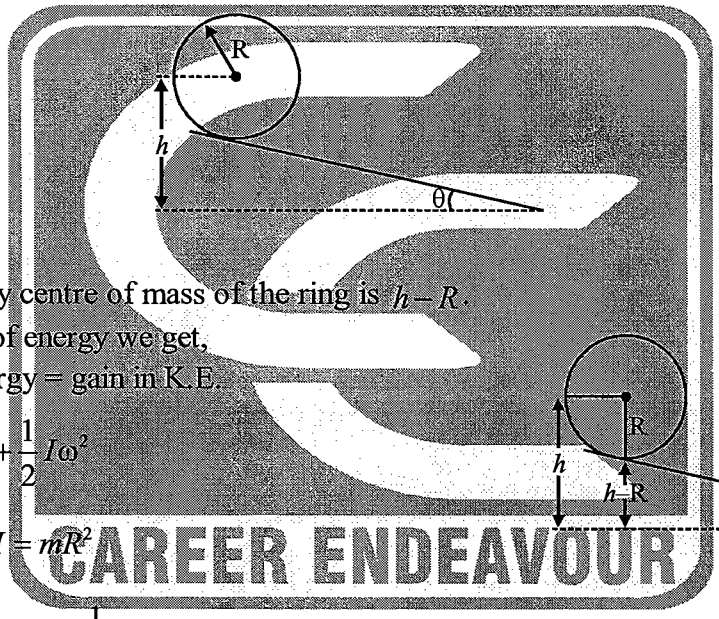


for rolling $\omega = v/r$, for hollow sphere $I = \frac{2}{3}mr^2$

$$\therefore\ mgh = \frac{1}{2}mv^2 + \frac{1}{2}.\frac{2}{3}mr^2.\frac{v^2}{r^2} \Rightarrow mgh = \frac{5}{6}mv^2$$

$$\therefore v = \sqrt{\frac{6gh}{5}}$$

Example: A ring of mass *m* and radius *R* rolls (without slipping) down an inclined plane starting from rest. If the centre of the ring is initially at a height *h*, the angular velocity when the ring reaches the base is

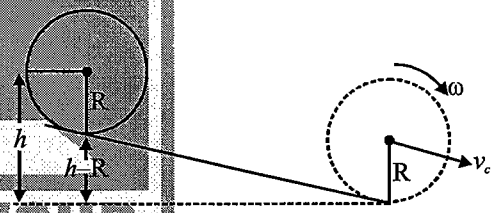


Soln. Height descended by centre of mass of the ring is $h-R$.

From conservation of energy we get,

loss in potential energy = gain in K.E.

$$mg(h-R) = \frac{1}{2}mv_c^2 + \frac{1}{2}I\omega^2$$



for rolling $v_c = \omega R, I = mR^2$

$$\therefore\ mg(h-R) = \frac{1}{2}m\omega^2R^2 + \frac{1}{2}mR^2\omega^2 \Rightarrow$$

$$mg(h-R) = m\omega^2R^2, \quad \therefore \omega = \sqrt{\frac{g(h-R)}{R^2}}$$

Acceleration of rolling object on inclined plane: Let us consider an object of mass M, radius R and moment of inertia I about the axis about which it rotates while rolling.

Let *a* and α be linear and angular acceleration of the object. Therefore for rolling

$a = \alpha R$ ... (i)

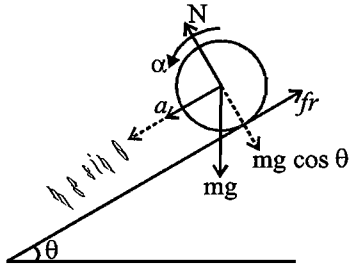
Equation of translational motion $F = ma$

or $mg\sin\theta - f_r = ma$ ... (ii)

Equation of rotational motion $\tau = I\alpha$

or $f_r R = I.\alpha$

or $f_r = \frac{Ia}{R^2}$ ... (iii)



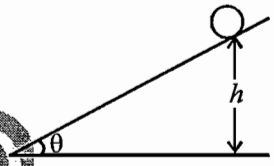
(ii) + (iii) gives $mg\sin\theta = a\left(m+\frac{I}{R^2}\right)$ or $\boxed{a=\frac{g\sin\theta}{1+\frac{I}{mR^2}}}$

from (iii) we get $\boxed{f_r=\frac{I}{R^2}.\frac{g\sin\theta}{1+\frac{I}{mR^2}}}$

friction force in rolling motion is static in nature. Therefore if μ be coefficient of static friction then

$f_r \le \mu N$ or $\frac{I}{R^2}.\frac{g\sin\theta}{1+\frac{I}{mR^2}} \le \mu.mg\cos\theta$ or $\boxed{\mu \ge \frac{\tan\theta}{1+\frac{mR^2}{I}}}$

Example: A solid sphere rolls down on inclined plane from a height 'h' as shown in figure. Calculate time taken by sphere to reach the bottom. Also calculate the minimum value of coefficient of friction necessary for rolling.



Soln. Moment of inertia of solid sphere $I=\frac{2}{5}mr^2$

acceleration of sphere $a=\frac{g\sin\theta}{1+\frac{I}{mr^2}}=\frac{g\sin\theta}{1+\frac{\frac{2}{5}mr^2}{mr^2}}=\frac{5g\sin\theta}{7}$

distance covered along the inclined plane $=\frac{h}{\sin\theta}$

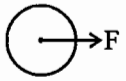
$\therefore$ If t be time taken then, $S=\frac{1}{2}at^2$

or $\frac{h}{\sin\theta}=\frac{1}{2}.\frac{5g\sin\theta}{7}t^2 \quad \because t=\sqrt{\frac{14h}{5g\sin^2\theta}}$

minimum value of coefficient of friction necessary to support rolling is

$$\mu=\frac{\tan\theta}{1+\frac{mr^2}{I}}=\frac{\tan\theta}{1+\frac{mr^2}{\frac{2}{5}mr^2}}=\frac{2}{7}\tan\theta$$

Example: A constant force F is applied horizontally at the centre of a circular disc of radius R and mass M as shown in the figure. If the disc rolls on the surface without sliding then. Calculate its acceleration.

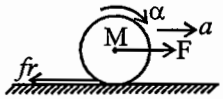


Soln. There must be friction between the disc and surface to support rolling. Let fr be force of friction. Therefore, translational equation of motion. $\vec{F}_{net}=m\vec{a}$

$f - f_r = Ma$ ... (a)

Rotational equation of motion $\tau = I.\alpha$

$$f_r R = \frac{2}{3}MR^2.\alpha$$



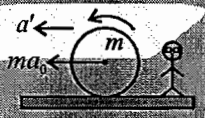
for pure rolling $a = \alpha R$

$$\therefore \quad f_r R = \frac{2}{3}MR^2.\frac{a}{R} \qquad \text{or} \quad f_r = \frac{2}{3}Ma \qquad \text{... (b)}$$

(a) + (b) gives, $F = \frac{5}{3}Ma \qquad \therefore \; a = \frac{3F}{5M}$

Example: A hollow sphere is kept on a plank which moves horizontally as shown in the figure. The sphere rolls on the plank. Calculate acceleration of the sphere with respect to ground.

Soln. Let us first find acceleration of sphere with respect to the plank. For this we observe motion of sphere from the plank's frame. Therefore a pseudo force has to be applied on the sphere. Now this is exactly the same problem as the previous one with applied force being equal to ma_0 i.e. $F = ma_0$. Therefore acceleration of sphere with respect to the plank will be



$$a' = \frac{3F}{5m} = \frac{3ma_0}{5m} = \frac{3}{5}a_0$$

Therefore acceleration of sphere with respect to the ground is $a = a_0 - \frac{3}{5}a_0 = \frac{2}{5}a_0$

Rotational about a fixed point: The formulae for angular momentum and kinetic energy $L = I\omega$ and kinetic energy $= \frac{1}{2}I\omega^2$ can be used if rotation is taking place about a fixed axis. In fact in some cases $L = I\omega$ represents angular momentum about the axis only, it is not equal to total angular momentum. One such situation is shown in figure below. In the figure we have shown two different rotational motion of a thin rod about a fixed axis

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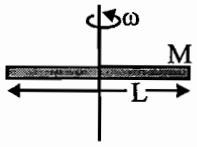
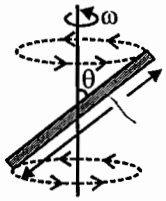


Angular momentum about the axis =

$$I\omega = \frac{ML^2}{12}\omega = \text{Total angular momentum}$$

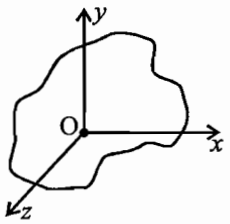


Angular momentum about the axis

$$= I\omega = \frac{ML^2}{12}\sin^2\theta\omega \neq \text{total angular momentum}$$

However, if rotation is taking place about a fixed point (not about fixed axis) then above mentioned formulae for angular momentum and kinetic energy can not be used. Let O be a fixed point about which a rigid body is rotating. Let us take a set of three perpendicular axes through O as shown in the figure. The axes taken are supposed to be fixed with the rigid body. Therefore w.r.t. to lab the orientation of axes keep changing. These are called body axes and are taken as instantaneous axes of rotation. If ω_x, ω_y and ω_z be components

of angular velocity and L_x, L_y and L_z components of angular momentum at some instant of time then value of angular momentum is calculated from following matrix relation.



$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

or $\boxed{\vec{L} = \vec{\vec{I}}.\vec{\omega}}$

and rotational kinetic energy is written as

$$K = \frac{1}{2}(\omega_x \omega_y \omega_z) \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \quad \text{or} \quad \boxed{K = \frac{1}{2}\vec{\omega}.\vec{\vec{I}}.\vec{\omega}} \quad \therefore \boxed{K = \frac{1}{2}\vec{\omega}.\vec{L}}$$

where $\vec{\vec{I}} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$ = inertia tensor

The element of inertia tensor are written as

Elements		Discrete case	Continuous case
Moment of inertia	$I_{xx} =$	$\sum_i m_i(y_i^2 + z_i^2)$	$\int (y^2 + z^2)dm$
	$I_{yy} =$	$\sum_i m_i(x_i^2 + z_i^2)$	$\int (x^2 + z^2)dm$
	$I_{zz} =$	$\sum_i m_i(x_i^2 + y_i^2)$	$\int (x^2 + y^2)dm$

Elements		Discrete case	Continuous case
Product of inertia	$I_{xy} = I_{yx} =$	$-\sum_i m_i x_i y_i$	$-\int xydm$
	$I_{xz} = I_{zx} =$	$-\sum_i m_i x_i z_i$	$-\int xzdm$
	$I_{yz} = I_{zy} =$	$-\sum_i m_i y_i z_i$	$-\int yzdm$

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Principal axes: A set of three axes so chosen that all the product of inertia becomes zero, are called principal axes. Therefore inertia tensor becomes diagonal if axes chosen happen to be principal axes.

Thus, if x, y, z are principal axes then $I_{ij} = 0$ if $i \neq j$

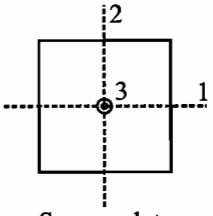
Therefore, $\vec{\vec{I}} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$

In this case I_{xx}, I_{yy}, I_{zz} are called principal moment of inertia. Thus principal moment of inertia are moment of inertia about principal axes. Principal axes are usually denoted by numbers 1, 2 and 3. Accordingly principal moment of inertia are denoted as I_1, I_2 and I_3.

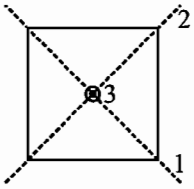
- Symmetric axes of an object are always principal axes.
- Principal axes may or may not be symmetric axes.

- Through each point of an object a set of three principal axes can be defined. Therefore an object can have infinite number of principal axes.

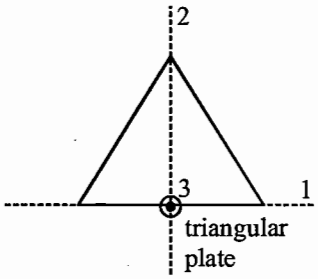
In the figure below principal axes of some objects are shown



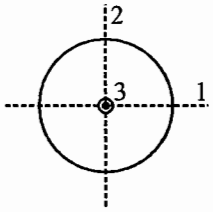
Square plate



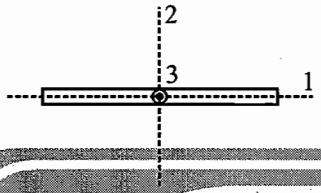
Square plate



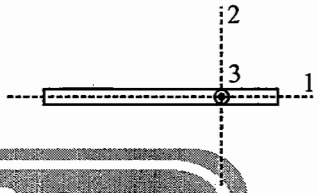
triangular plate



Sphere



Thin rod

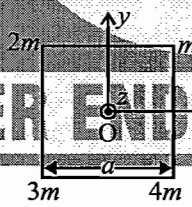


Thin rod

Finding principal moment of inertia and principal axes: Eigen values of inertia tensor are equal to principal moment of inertia and eigen vectors of inertia tensor give orientation of principal axes.

Therefore, to find principal axes through a point of an asymmetric object we can choose any three convenient axes and write inertia tensor and to get principal axes we need to calculate eigen vectors.

Example: Four point masses $m, 2m, 3m$ and $4m$ are placed at the corners of a square of side 'a' as shown in the figure. Write inertia tensor for the axes shown in figure and find the principal axes passing through point O.



CAREER ENDEAVOUR

Soln. Coordinates of masses are, $m\left(\frac{a}{2},\frac{a}{2},0\right)$, $2m\left(\frac{-a}{2},\frac{a}{2},0\right)$, $3m\left(\frac{-a}{2},\frac{-a}{2},0\right)$, $4m\left(\frac{a}{2},\frac{-a}{2},0\right)$

therefore, $I_{xx}=\sum_i m_i(y_i^2+z_i^2)=m\left(\frac{a^2}{4}\right)+2m\left(\frac{a^2}{4}\right)+3m\left(\frac{a^2}{4}\right)+4m\left(\frac{a^2}{4}\right)=\frac{5}{2}ma^2$

$I_{yy}=\sum m_i(x_i^2+z_i^2)=m\left(\frac{a^2}{4}\right)+2m\left(\frac{a^2}{4}\right)+3m\left(\frac{a^2}{4}\right)+4m\left(\frac{a^2}{4}\right)=\frac{5}{2}ma^2$

$I_{zz}=\sum m_i(x_i^2+y_i^2)=5ma^2$

$I_{xy}=-\sum m_i x_i y_i=-m\left(\frac{a}{2}\right).\left(\frac{a}{2}\right)-2m\left(\frac{-a}{2}\right).\left(\frac{a}{2}\right)-3m\left(\frac{-a}{2}\right)\left(\frac{-a}{2}\right)-4m\left(\frac{a}{2}\right)\left(\frac{-a}{2}\right)$

$=\frac{ma^2}{4}(-1+2-3+4)=\frac{ma^2}{2}=I_{yx}$

$$I_{xz} = I_{zx} = -\sum m_i x_i z_i = 0 \ (\because z_i = 0), \quad I_{yz} = I_{zy} = -\sum m_i y_i z_i = 0 \ (\because z_i = 0)$$

therefore, $II = ma^2 \begin{pmatrix} 5/2 & 1/2 & 0 \\ 1/2 & 5/2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

Principal moment of inertia: Let λ be eigen value of inertia tensor.

$$\therefore \begin{vmatrix} \frac{5ma^2}{2} - \lambda & \frac{ma^2}{2} & 0 \\ \frac{ma^2}{2} & \frac{5ma^2}{2} - \lambda & 0 \\ 0 & 0 & \frac{5ma^2}{2} - \lambda \end{vmatrix} = 0 \qquad \therefore \left(\frac{5ma^2}{2} - \lambda\right)\left[\left(\frac{5ma^2}{2} - \lambda\right)^2 - \left(\frac{ma^2}{2}\right)^2\right] = 0$$

$$\therefore \ \frac{5ma^2}{2} - \lambda = 0 \quad \text{or} \quad \left(\frac{5ma^2}{2} - \lambda\right)^2 - \left(\frac{ma^2}{2}\right)^2 = 0$$

$$\therefore \ \lambda = \frac{5ma^2}{2} \quad \text{or} \quad \frac{5ma^2}{2} - \lambda = \pm\frac{ma^2}{2} \Rightarrow \lambda = 2ma^2, 3ma^2$$

$\therefore \ \lambda = 2ma^2, 3ma^2, 5ma^2$ (Principal moment of inertia). Let $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ be eigen vector corresponding to eigen value $2ma^2$.

Therefore,

$$\begin{pmatrix} \frac{5ma^2}{2} & \frac{ma^2}{2} & 0 \\ \frac{ma^2}{2} & \frac{5ma^2}{2} & 0 \\ 0 & 0 & 5ma^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 2ma^2 \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \quad \begin{pmatrix} \frac{5ma^2}{2}\alpha + \frac{ma^2}{2}\beta \\ \frac{ma^2}{2}\alpha + \frac{5ma^2}{2}\beta \\ 5ma^2\gamma \end{pmatrix} = \begin{pmatrix} 2ma^2\alpha \\ 2ma^2\beta \\ 2ma^2\gamma \end{pmatrix}$$

$\therefore \quad 5\alpha + \beta = 4\alpha \quad$ or $\quad \alpha = -\beta$

If $\alpha = 1$ then $\beta = -1$

Therefore, normalized eigen vector corresponding to $\lambda = 2ma^2$ is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

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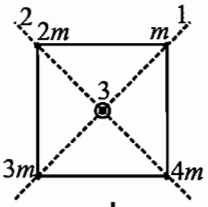
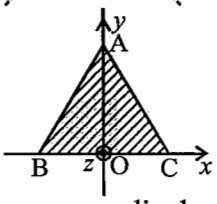
Similarly we find that normalized eigen vector corresponding to $\lambda = 3ma^2$ is $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$.

Thus principal axes for given mass distribution are as shown in figure:



Example: A uniform equilateral triangular plate of mass 'M' and side 'a' lies in x-y plane as shown in figure. Write inertia tensor.



Soln. y-axis is a symmetric axis therefore it must be a principal axis. And x-being perpendicular to y, is also a principal axis.

Therefore, $x-y-z$ axes given in figure are principal axes. Therefore all product of inertia are zero.

And, $I_{xx} = \int(y^2+z^2)dm = \int y^2 dm \quad (\because z=0)$

$$= \iint y^2 . \frac{M}{\frac{\sqrt{3}}{4}a^2} . dxdy = \frac{4M}{\sqrt{3}a^2}\int dx \int y^2 dy$$

$$= \frac{8M}{\sqrt{3}a^2}\int_0^{a/2} dx \int_0^{\frac{\sqrt{3}}{2}(a-2x)} y^2 dy \qquad \{\because \text{equation of AC is } y = -\sqrt{3}x + \frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{2}(a-2x)\}$$

$$= \frac{M}{a^2}\int_0^{a/2} dx(a-2x)^3 = \frac{M}{a^2}\left[-\frac{1}{2}\frac{(a-2x)^4}{4}\right]_0^{a/2} = \frac{Ma^2}{8}$$

CAREER ENDEAVOUR

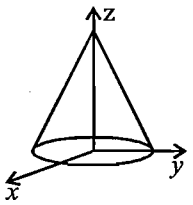
$$I_{yy} = \int(x^2+z^2)dm = \int x^2 dm = \iint x^2 . \frac{M}{\frac{\sqrt{3}}{4}a^2} dx.dy = \frac{4M}{\sqrt{3}a^2}\iint x^2 dxdy = \frac{8M}{\sqrt{3}a^2}\int_0^{a/2} x^2 dx \int_0^{\frac{\sqrt{3}}{2}(a-2x)} dy$$

$$= \frac{4M}{a^2}\int_0^{a/2} x^2(a-2x)dx = \frac{4M}{a^2}\left[a.\frac{\left(\frac{a}{2}\right)^3}{3} - 2.\frac{\left(\frac{a}{2}\right)^4}{4}\right] = \frac{4M}{a^2}\left[\frac{a^4}{24} - \frac{a^4}{32}\right] = \frac{Ma^2}{24}$$

$$I_{zz} = \int(x^2+y^2)dm = I_{xx} + I_{yy} = \frac{Ma^2}{8} + \frac{Ma^2}{24} = \frac{Ma^2}{6}$$

Therefore, $II = Ma^2\begin{pmatrix}1/8 & 0 & 0\\ 0 & 1/24 & 0\\ 0 & 0 & 1/6\end{pmatrix}$

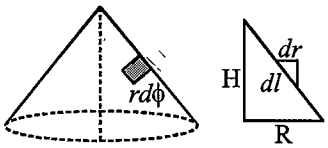
Example: A hollow right circular cone has mass M, radius R, height H. Find inertia tensor of the cone about the axes shown in the figure.



Soln. x, y, z are symmetric axes. Therefore, these are principal axes.
So the inertia tensor is diagonal i.e. all product of inertia must be zero.
let us use cylindrical coordinate system

$\therefore \quad x = r\cos\phi, y = r\sin\phi, z = z$

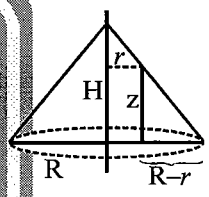
Area element on the surface of cone = $rd\phi dl = rd\phi.\dfrac{\sqrt{R^2+H^2}}{R}dr$



Area of slanting surface of cone = $\pi R\sqrt{R^2+H^2}$

$$\therefore \quad dm = \frac{M}{\pi R\sqrt{H^2+R^2}}.rd\phi\frac{\sqrt{R^2+H^2}}{R}dr = \frac{M}{\pi R^2}.rdrd\phi$$

$$I_{xx} = \int(y^2+z^2)dm = \iint(r^2\sin^2\phi+z^2).\frac{M}{\pi R^2}rdrd\phi$$



Consider a point on surface of cone as shown in figure
using similarity of triangles we get

$$\frac{z}{R-r} = \frac{H}{R} \quad \therefore \quad z = H\left(1-\frac{r}{R}\right)$$

$$\therefore I_{xx} = \iint\left[r^2\sin^2\phi + H^2\left(1-\frac{r}{R}\right)^2\right]\frac{M}{\pi R^2}.rdrd\phi$$

$$= \frac{M}{\pi R^2}\left[\int_0^R r^3dr\int_0^{2\pi}\sin^2\phi d\phi + H^2\int_0^R\left(1-\frac{r}{R}\right)^2 rdr\int_0^{2\pi}d\phi\right] = \frac{M}{\pi R^2}\left[\frac{R^4}{4}.\pi + H^2\frac{R^2}{12}.2\pi\right] = \frac{M}{4}\left[R^2+\frac{2H^2}{3}\right]$$

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x and y axes are equivalent. Therefore,

$$I_{yy} = \frac{M}{4}\left(R^2+\frac{2H^2}{3}\right)$$

$$I_{zz} = \int(x^2+y^2)dm = \iint(r^2\cos^2\phi+r^2\sin^2\phi).\frac{M}{\pi R^2}rdrd\phi$$

$$= \frac{M}{\pi R^2}\int_0^R r^3dr\int_0^{2\pi}d\phi = \frac{MR^2}{2} \qquad \therefore II = \begin{pmatrix} \frac{M}{4}\left(R^2+\frac{2H^2}{3}\right) & 0 & 0 \\ 0 & \frac{M}{4}\left(R^2+\frac{2H^2}{3}\right) & 0 \\ 0 & 0 & \frac{MR^2}{2} \end{pmatrix}$$

Example: Inertia tensor of a system is $\begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix}$. Calculate moment of inertia about an axis which is oriented in the direction $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

Soln. Given $II = \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix}$ and $\hat{n} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\therefore \quad I = \hat{n}.\vec{\vec{II}}.\hat{n}$

$$= \frac{1}{3}(1 \quad 1 \quad 1)\begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{3}(1 \quad 1 \quad 1)\begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \frac{1}{18}(1 \quad 1 \quad 1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{18}(1+1+1) = \frac{1}{6}$$

CAREER ENDEAVOUR

Example: Four equal point masses m are placed the corners of a rectangle of sides 'a' and 'b'. The system is rotated about a diagonal with uniform angular velocity ω. Calculate rotational kinetic energy, angular momentum about the axis of rotation and total angular momentum of system.

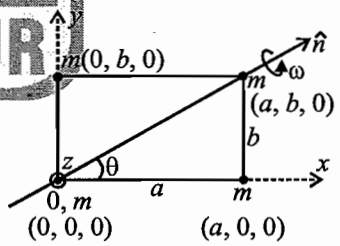
Soln. Let us choose x-y-z axes as shown in figure.

$$I_{xx} = \sum m_i(y_i^2 + z_i^2) = \sum m_i y_i^2 \quad (\because z_i = 0)$$

$$= m0^2 + m0^2 + mb^2 + mb^2 = 2mb^2$$

$$I_{yy} = \sum_i m_i(x_i^2 + z_i^2) = \sum m_i x_i^2 = m0^2 + ma^2 + m0^2 + ma^2 = 2ma^2$$

$$I_{zz} = \sum_i m_i(x_i^2 + y_i^2) = 2m(a^2 + b^2)$$



$$I_{xy} = -\sum m_i x_i y_i = -mab = I_{yx} \qquad \therefore \; II = m\begin{pmatrix} 2b^2 & -ab & 0 \\ -ab & 2a^2 & 0 \\ 0 & 0 & 2(a^2+b^2) \end{pmatrix}$$

Direction of diagonal,

$$\hat{n} = \cos\theta\hat{i} + \sin\theta\hat{j} = \frac{1}{\sqrt{a^2+b^2}}(a\hat{i} + b\hat{j}) \equiv \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

moment of inertia about the axis of rotation

$$I = \hat{n}.\vec{\vec{II}}.\hat{n} = \frac{m}{(a^2+b^2)}(a \quad b \quad 0)\begin{pmatrix} 2b^2 & -ab & 0 \\ -ab & 2a^2 & 0 \\ 0 & 0 & 2(a^2+b^2) \end{pmatrix}\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

$$= \frac{m}{a^2+b^2}(a \quad b \quad 0)\begin{pmatrix} ab^2 \\ a^2b \\ 0 \end{pmatrix} = \frac{2ma^2b^2}{(a^2+b^2)}$$

Therefore, rotational kinetic energy $= \frac{1}{2}I\omega^2 = \frac{1}{2}\frac{2ma^2b^2}{(a^2+b^2)}\omega^2 = \frac{ma^2b^2\omega^2}{a^2+b^2}$

Angular momentum about the axis of rotation $= I\omega = \frac{2ma^2b^2\omega}{(a^2+b^2)}$

To calculate total angular momentum of system we use $\vec{L} = \vec{I}.\vec{\omega}$

or $\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$, Here $\omega_x = \omega\cos\theta = \frac{\omega a}{\sqrt{a^2+b^2}}$, $\omega_y = \omega\sin\theta = \frac{\omega b}{\sqrt{a^2+b^2}}$, $\omega_z = 0$

$$\therefore \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \frac{m\omega}{\sqrt{a^2+b^2}}\begin{pmatrix} 2b^2 & -ab & 0 \\ -ab & 2a^2 & 0 \\ 0 & 0 & 2(a^2+b^2) \end{pmatrix}\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \frac{m\omega}{\sqrt{a^2+b^2}}\begin{pmatrix} ab^2 \\ a^2b \\ 0 \end{pmatrix}$$

$$\therefore L_x = \frac{m\omega ab^2}{\sqrt{a^2+b^2}},\ L_y = \frac{m\omega a^2b}{\sqrt{a^2+b^2}},\ L_z = 0$$

$$\therefore L = \sqrt{L_x^2+L_y^2+L_z^2} = \frac{m\omega ab}{\sqrt{a^2+b^2}}\sqrt{a^2+b^2}$$

$$\therefore L = m\omega ab$$

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"Make Physics as simple as possible but not simpler"

Albert Einstein

SPECIAL THEORY OF RELATIVITY (STR)

In 1905 Albert Einstein published a paper with a title "On the electrodynamics of moving bodies". This work of Einstein gave a totally different notion of Physical world. It changed the foundation of Physics and also our understanding of nature.

This theory has originated from the analysis of electromagntic theory (Maxwell's Equation). In most of the books discussion of special theory of relativity starts with Michelsion Morley experiment. Therefore, it appears as if this theory came into existence in search of explanation for negative result of Michelson Morley experiment. But that is is not true. Einstein was not even aware of Michelson Morley experiment.

In the following we briefly discuss the fundamental concepts of this theory.

Frame of reference: A place from where observations are made is called frame of reference. In Physics a frame of reference is represented by a co-ordinate system with respect to which spatial co-ordinate of any point in space is determined and we also associate a clock with the coordinate system to measure time of any event in space.

(Frame of reference)

Frame of reference are of two types:

1. **Inertial frame of reference:** Two frames are said to be inerteial if they are at rest or moving with constant velocity w.r.t each other.

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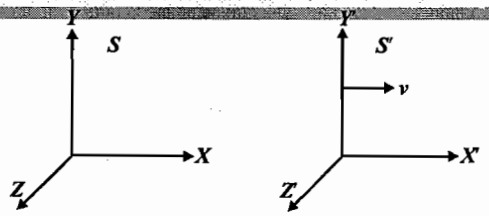
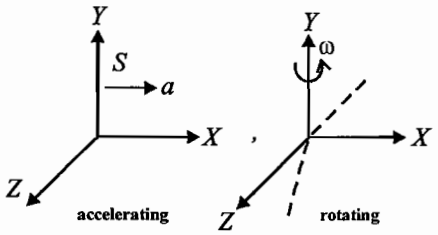


In figure shown S' is moving with constant speed v. w.r.t S, therefore, S and S' are inertial frames.

2. **Non-inertial frame of reference:** A frame of reference is said to be non-inertial if it is either accelerating or rotating



Frame shown in the figure above are non-inertial.

Note: We define 'two' inertial frames whereas we define single non-inertial frame. Because absolute (single) inertial frame doesn't exist. Whereas single (absolute) non-inertial frame exists.

Postulates of STR:

STR deals with inertial frames of reference. And it is based on following two postulates.

First Postulate:

All inertial frames are 'equivalent'. Therefore, laws of physics must be invariant in all inertial frames.

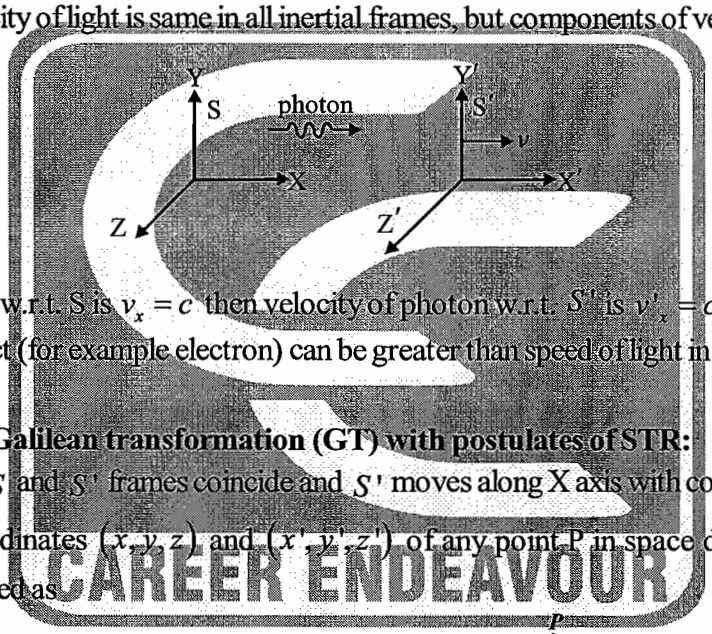
What this postulate implies is that, if an experiment is done in laboratory (Neglect earth's motion) and the same experiment is repeated in a van moving with constant velocity then result of experiment in the two cases will be same.

Second Postulate:

Speed (magnitude) of light in vacuum has same value in all inertial frame. And speed of light is the ultimate speed in vacuum.

What this postulate implies is that,

* Speed of any object or anything (which is real/observable) cannot be greater than speed of light in vacuum.
* Magnitude of velocity of light is same in all inertial frames, but components of velocity of light changes.



If velocity of photon w.r.t. S is $v_x = c$ then velocity of photon w.r.t. S' is $v'_x = c$ (i.e. same as in S frame)

* Velocity of an object (for example electron) can be greater than speed of light in some medium (for example water).

Incompatibility of Galilean transformation (GT) with postulates of STR:

If at $t = 0$, origins of S and S' frames coincide and S' moves along X axis with constant velocity 'v' as shown in figure, then co-ordinates (x, y, z) and (x', y', z') of any point P in space determined from S and S' respectively are related as

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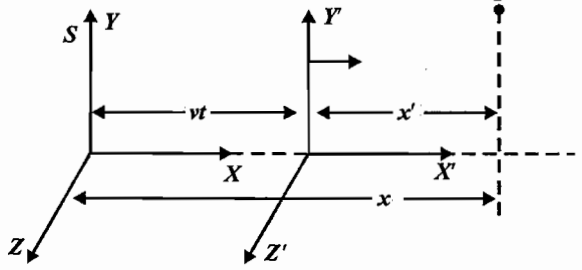

$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \end{aligned}\right\} \text{ G-T for co-ordinats} \qquad \text{.... (1.1)}$$

Differentiating w.r.t time we get

$$\left.\begin{aligned} v'_x &= v_x - v \\ v'_y &= v_y \\ v'_z &= v_z \end{aligned}\right\} \text{ G-T for velocity} \qquad \text{... (1.2)}$$

Here v_x, v_y and v_z are components of velocity of point P in S frame and v'_x, v'_y and v'_z are components of velocity in S' frame. [Point P can be any objects an insect, a car, a photon etc]

If P be a photon and let's suppose it is moving along x-axis with speed 'c' w.r.t S frame then $v_x = c$. Therefore according to GT speed of photon w.r.t S' frame is

$$v'_x = v_x - v = c - v \quad \text{(i.e. different from that in S frame)}$$

Thus, we see that GT doesn't give result in accordance with postulates of STR. Therefore, it is not correct and must be modified.

About G-T we also note following points

- On differentiating velocity relation (1.2) we get $a'_x = a_x, a'_y = a_y, a'_z = a_z$ thus acceleration is invariant under G-T.
- From co-ordinate relation (1.1) at any instant of time, we get $\Delta x' = \Delta x$, i.e. distance (length) between two points is same in S and S'. Therefore length of an object is invariant under G-T.
- Time measured in S and S' is taken to be same i.e. $t' = t$. Thus in G-T it is assumed that time measurement is not affected by relative motion.
- Relation (1.1) is simply a geometrical relation. Therefore, incompatibility of G-T with S.T.R tells that, due to relative motion geometry must change or physical property of objects will change due to relative motion.

Lorentz Transformation (L-T):

Einstein modified the G-T to make it compatible with postulates of STR. The modified relations are called Lorentz transformation after the name of German Mathematician Lorentz who had obtained the same transformation relations before Einstein. But Lorentz's derivation was not based on any concrete physical argument.

L-T for co-ordinates is given as

$$\left.\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}\right\} \quad \text{L-T for co-ordinates} \qquad \ldots (1.3)$$

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By using two postulates of S.T.R. Einstein showed that $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ (v = velocity of S' frame w.r.t. S frame)

and also showed that

$$t' = \gamma\left(t - \frac{xv}{c^2}\right) \qquad \text{L-T for time} \qquad \ldots (1.4)$$

Relations 1.3 and 1.4 are used when co-ordinates and time of an event in S frame are given and values in S' are to be found.

If values are known in S' and values in S frame are to be found then we use inverse Lorentz transformation.

$$\left.\begin{aligned} x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{x'v}{c^2}\right) \end{aligned}\right\} \quad \text{inverse L-T} \qquad \ldots (1.5)$$

Note that transformation relation (1.3) to (1.5) have been written for the case when S' is moving along X axis

w.r.t. S. If S' moves along Y w.r.t. S then we replace x by y and y by x every where i.e.

$$\left.\begin{aligned} x' &= x \\ y' &= \gamma(y - vt) \\ z' &= z \\ t' &= \gamma\left(t - \frac{yv}{c^2}\right) \end{aligned}\right\}$$ L-T when S' moving along y direction w.r.t. S. ... (1.6)

Differential form of Lorentz transformation:

Taking differential of equation (1.3) and (1.4) we get

$$\left.\begin{aligned} \Delta x' &= \frac{\Delta x - v\Delta t}{\sqrt{1 - v^2/c^2}} \\ \Delta y' &= \Delta y \\ \Delta z' &= \Delta y \\ \Delta t' &= \frac{\Delta t - \Delta x.v/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned}\right\}$$ ... (1.7)

These relations can be used to find time difference or spatial separation between two events.

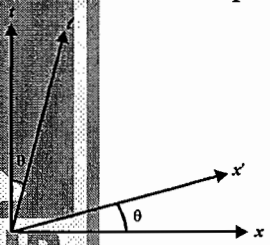
L-T as fictitious rotation in spacetime:

L-T can be thought of as a fictitious rotation of x, t axes. Let $\frac{v}{c} = \tanh\theta$, therefore from equation (1.3) and (1.4) we get,

$$x' = x\cosh\theta - ct\sinh\theta$$

$$ct' = ct\cosh\theta - x\sinh\theta$$

Geometrically we can obtain above relations if we consider rotation of x to x' and t to t' as shown in figure.



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Example: Two events separated by a spatial distance 3×10^9 m are simultaneous in one inertial frame. Find the time interval between these two events in an inertial frame moving with a constant speed 0.6c.

Soln. Let $\Delta x = 3\times10^9\ m$ $\quad\Delta t = 0$

$v = 0.6c,$ $\quad\Delta t' = ?$

From differential form of L-T

$$\Delta t' = \frac{\Delta t - \frac{\Delta x v}{c^2}}{\sqrt{1 - v^2/c^2}} = \frac{0 - \frac{3\times10^9 \times 0.6\times3\times10^8}{\left(3\times10^8\right)^2}}{\sqrt{1 - 0.36}} = -28.125$$

Therefore time interval between two events in moving frame is 28.125 sec.

Example: In an inertial frame S an event takes place at (200, 0, 0) and 2×10^{-6} second later another event takes place at (1000, 0, 0). Find the magnitude and direction of velocity of a frame S' in which these events appear simultaneous.

Soln. Given, $\Delta x = 800m,$ $\quad\Delta t = 2\times10^{-6}$ sec, $\Delta t' = 0$

From differential form of L-T, $\Delta t' = \dfrac{\Delta t - \dfrac{\Delta x v}{c^2}}{\sqrt{1 - v^2/c^2}}$

Therefore, $0 = \left(2\times10^{-6} - 800\, v/c^2\right)/\sqrt{1 - v^2/c^2}$

Or, $v = \dfrac{2\times10^{-6}}{800}\times 3\times10^8 c \qquad \Rightarrow v = \dfrac{3}{4}c$

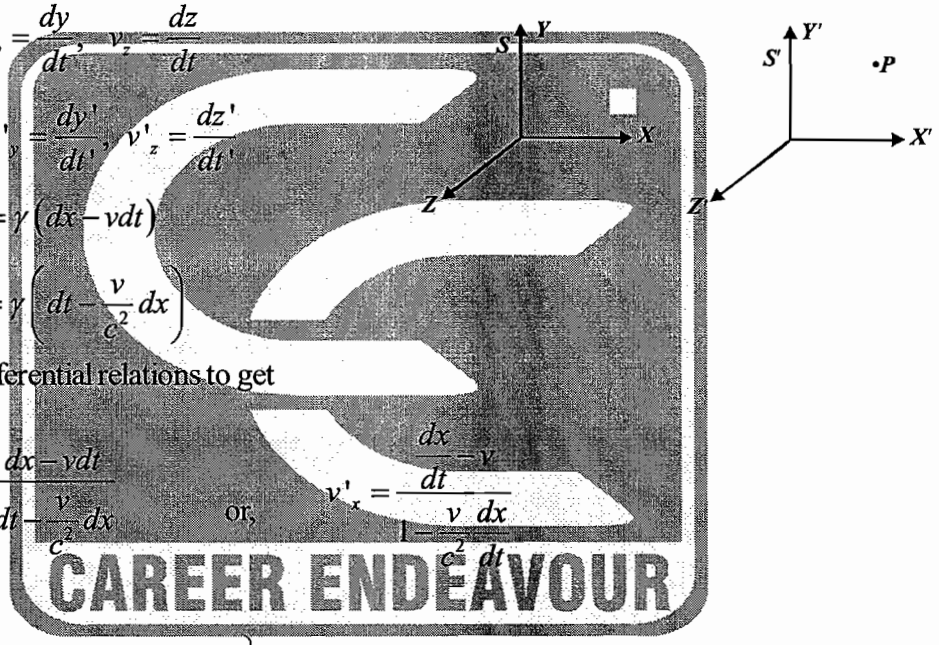
Velocity obtained is positive. Therefore, the frame should move in the direction from point (200, 0, 0) i.e. along $+x$ direction.

Velocity Transformation:

Let v_x, v_y and v_z be velocity of components of P in S frame and v'_x, v'_y and v'_z be velocity components of P in S'.

$\therefore \quad v_x = \dfrac{dx}{dt}, \quad v_y = \dfrac{dy}{dt}, \quad v_z = \dfrac{dz}{dt}$

$v'_x = \dfrac{dx'}{dt'}, \quad v'_y = \dfrac{dy'}{dt'}, \quad v'_z = \dfrac{dz'}{dt'}$



from (1.3) $dx' = \gamma\left(dx - v dt\right)$

from (1.4) $dt' = \gamma\left(dt - \dfrac{v}{c^2}dx\right)$

Divide above two differential relations to get

$$\frac{dx'}{dt'} = v'_x = \frac{dx - v dt}{dt - \frac{v}{c^2}dx} \qquad \text{or,} \qquad v'_x = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}}$$

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$$\left.\begin{aligned} \text{therefore} \quad & v'_x = \frac{v_x - v}{1 - v v_x/c^2} \\ \text{similarly,} \quad & v'_y = \frac{v_y\sqrt{1 - \frac{v^2}{c^2}}}{1 - v v_x/c^2} \\ \text{and} \quad & v'_z = \frac{v_z\sqrt{1 - \frac{v^2}{c^2}}}{1 - v v_x/c^2} \end{aligned}\right\}$$ L-T for velocity ... (1.8)

Relation 1.8 are used to find velocity w.r.t. S' when velocity w.r.to S is given.

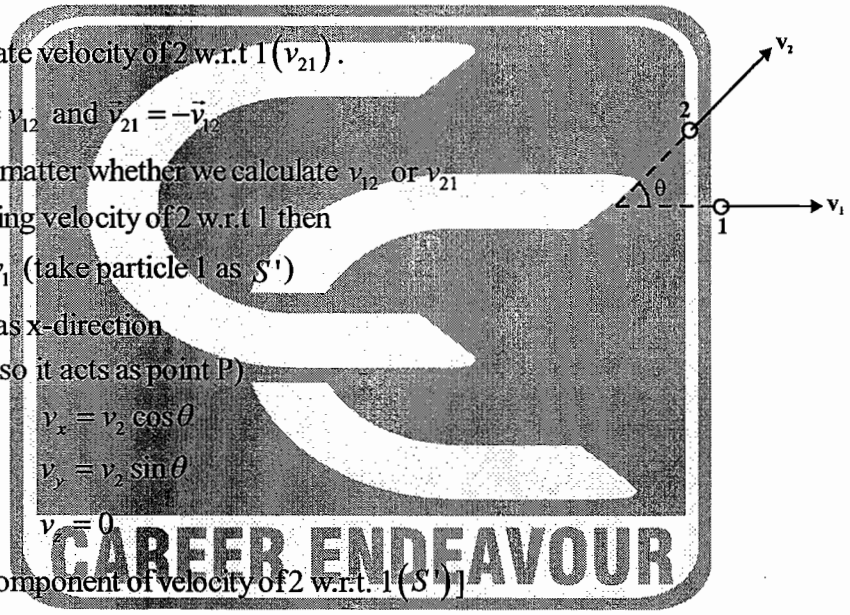
If velocity w.r.t S' is given and velocity w.r.t. S is to be found then we use inverse L-T for velocity which are given as

$$\left.\begin{aligned} v_x &= \frac{v'_x + v}{1 + vv'_x/c^2} \\ \text{similarly,}\quad v_y &= \frac{v'_y\sqrt{1-\frac{v^2}{c^2}}}{1 + vv'_x/c^2} \\ \text{and}\quad v_z &= \frac{v'_z\sqrt{1-\frac{v^2}{c^2}}}{1 + vv'_x/c^2} \end{aligned}\right\}$$ Inverse L-T for velocity ... (1.9)

Method to use velocity transformation formula: If velocity w.r.t some moving object is either given or is to be calculated, consider that object as S' frame. Thus v = velocity of moving object w.r.t which velocity is given or to be calcualted.

General relativistic formula for relative velocity between two particles:

Let v_1 and v_2 be speed of two particles in lab frame (S-frame) and θ be angle between their directions as shown in figure.



Let us calculate velocity of 2 w.r.t 1 (v_{21}).

Actually, $v_{21} = v_{12}$ and $\vec{v}_{21} = -\vec{v}_{12}$

Therefore, it doesn't matter whether we calculate v_{12} or v_{21}

So, if we are calculating velocity of 2 w.r.t 1 then

$v = v_1$ (take particle 1 as S')

Take direction of v_1 as x-direction

We are observing 2 (so it acts as point P)

$\therefore \quad v_x = v_2\cos\theta$

and $v_y = v_2\sin\theta$

$v_z = 0$

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using (1.8) we get [component of velocity of 2 w.r.t. 1 (S')]

$$v'_x = \frac{v_2\cos\theta - v_1}{1 - \frac{v_1.v_2}{c^2}\cos\theta}; \qquad v'_y = \frac{v_2\sin\theta\sqrt{1-v_1^2/c^2}}{c^2}; \qquad v'_z = 0$$

Therefore, resultant relative velocity v_{21}

$$= \sqrt{v_{x'}^2 + v_{y'}^2 + v_{z'}^2} = \frac{\sqrt{\left(v_2\cos\theta - v_1\right)^2 + v_2^2\sin^2\theta\left(1-\frac{v_1^2}{c^2}\right)}}{\left(1 - \frac{v_1.v_2\cos\theta}{c^2}\right)}$$

$$\boxed{v_{21} = \frac{\sqrt{v_1^2 + v_2^2 - 2v_1.v_2\cos\theta - \frac{v_1^2v_2^2\sin^2\theta}{c^2}}}{1 - \frac{v_1.v_2\cos\theta}{c^2}}} \qquad \text{... (1.10)}$$

(1.10) is an important formula which can be used to find relative velocity between two particles moving at any angle.

Example: A pion moving with speed 0.6c emits a muon in the same direction. Speed of muon in the rest frame of pion is 0.4c. Calculate the speed of muon in lab frame.

Soln. Velocity of pion w.r.t. lab (v) = 0.6c

Velocity of muon w.r.t. pion $v'_x = 0.4c$

Therefore, velocity muon in lab frame (use inverse L-T)

$$v_x = \frac{v'_x + v}{1 + \frac{v'_x . v}{c^2}} = \frac{0.4c + 0.6c}{1 + 0.24} = \frac{c}{1.24} = \frac{25c}{31}$$

Example: A person going in car which is moving w.r.t. gorund with speed $\frac{3c}{5}$ throws a ball in upward direction with speed $c/2$. What is the speed of the ball w.r.t ground.

Soln. Let us take car as S' frame and ground as S frame.

$\therefore$ $v = \frac{3c}{5}$

Component of velocity of ball in car's frame

$$v_x = 0, \; v'_y = \frac{c}{2} \quad \Rightarrow v_x = \frac{v'_x + v}{1 + \frac{v'_x . v}{c^2}} = \frac{0 + \frac{3c}{5}}{1 + 0} = \frac{3c}{5}$$

$$v_y = \frac{v'_y \sqrt{1 - v^2/c^2}}{1 + \frac{v'_x . v}{c^2}} = \frac{\frac{c}{2}\sqrt{1 - \frac{9}{25}}}{1 + 0} = \frac{2c}{5}$$

Velocity of ball as seen from ground.

$$= \sqrt{v_x^2 + v_y^2} = \sqrt{\frac{9}{25}c^2 + \frac{4}{25}c^2} = \frac{\sqrt{13}c}{5}$$

CAREER ENDEAVOUR

Example: A car is moving with respect to ground along +x direction with speed c/2. A ball is thrown from car with speed $c/\sqrt{2}$. As seen from ground, the ball moves in +y direction. Calculate speed of ball w.r.t. ground.

Soln. Velocity of car w.r.t. ground (v) = c/2

Velocity of ball w.r.t. grond along x direction $v_x = 0$

Let v'_x and v'_y be velocity components of ball w.r.t. car.

$\therefore$ $$v_x = \frac{v'_x + v}{1 + v'_x . v / c^2}$$

Or, $$0 = \frac{v'_x + v}{1 + \frac{v'_x . v}{c^2}} \quad \Rightarrow v'_x = -v = -c/2$$

Velocity of ball w.r.t. car = $c/\sqrt{2}$

$\therefore$ $$v'^2_x + v'^2_y = \left(\frac{c}{\sqrt{2}}\right)^2$$

$$v'^2_y = \frac{c^2}{2} - v'^2_x = \frac{c^2}{2} - \frac{c^2}{4} = \frac{c^2}{4}$$

$$v'_y = \frac{c}{\sqrt{2}}$$

y-component of velocity of ball w.r.t. ground.

$$v_y = \frac{v'_y\sqrt{1-v^2/c^2}}{1+\frac{v'x.v}{c^2}} = \frac{\frac{c}{\sqrt{2}}\sqrt{1-\frac{1}{4}}}{1-\frac{1}{4}} = \frac{c\sqrt{2}}{\sqrt{3}}$$

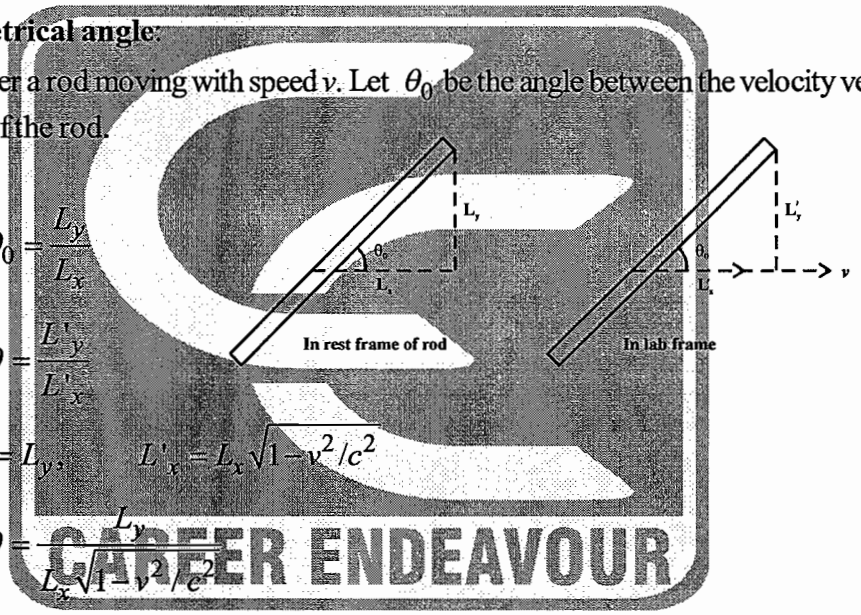
Therefore, velocity of ball w.r.t. ground

$$= \sqrt{v_x^2 + v_y^2} = \sqrt{0+\left(\frac{c\sqrt{2}}{\sqrt{3}}\right)^2} = \sqrt{\frac{2}{3}}c$$

Relativity of geometrical angle:

Let us consider a rod moving with speed v. Let θ_0 be the angle between the velocity vector and length of rod in rest frame of the rod.

From figure,



$$\tan\theta_0 = \frac{L_y}{L_x}$$

$$\tan\theta = \frac{L'_y}{L'_x}$$

and $L'_y = L_y$, $L'_x = L_x\sqrt{1-v^2/c^2}$

$$\therefore \quad \tan\theta = \frac{L_y}{L_x\sqrt{1-v^2/c^2}}$$

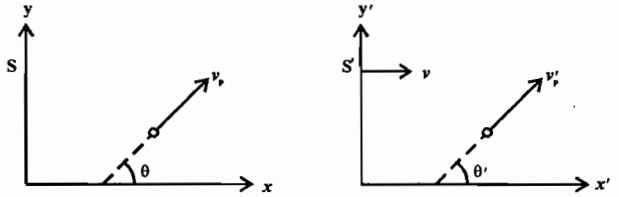
Or,
$$\boxed{\tan\theta = \frac{\tan\theta_0}{\sqrt{1-v^2/c^2}}} \qquad \text{... (1.11)}$$

As $\sqrt{1-v^2/c^2} < 1$ therefore $\theta > \theta_0$. Thus, geometrical angle made by an object with direction of velocity appears more in the frame in which object is moving than in the rest frame.

Relativity of angle made by velocity vector:

Let velocity of a particle makes an angle θ with +x direction in S frame and its velocity makes an angle θ' in S' frame.

Let v_p and v'_p be velocities of particle in S and S' frame and v_x and v'_x be x-components of velocities in the two frames respectively.



Therefore, angle made with x-direction in two frame are,

$$\cos\theta = \frac{v_x}{v_p}; \quad v_p = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$\cos\theta' = \frac{v'_x}{v'_p}; \quad v'_p = \sqrt{v'^2_x + v'^2_y + v'^2_z}$$

We can use velocity transformation relations to relate velocity in two frames. By using relation between velocities we can get relation between angle θ and θ'.

Example: A car is moving with speed $\frac{c}{2}$. A light beam is emitted from car at angle 45° with the direction of velocity of car. What is the angle with velocity vector as seen from ground.

Soln. Velocity of light is same in all inertial frames. Therefore,

$$v_p = v'_p = c \quad \text{(here instead of a particle we are looking at light beam)}$$

Take car as S' frame. Therefore,

$$v = \frac{c}{2}, \quad \theta' = 45°, \quad v'_x = v'_p \cos\theta' = \frac{c}{\sqrt{2}}$$

Using velocity trans formation we get,

$$v_x = \frac{v'_x + v}{1 + \frac{v'_x v}{c^2}} = \frac{\frac{c}{\sqrt{2}} + \frac{c}{2}}{1 + \frac{1}{2\sqrt{2}}} = \frac{\sqrt{2}+1}{1+2\sqrt{2}}c$$

Therefore, angle measured in ground frame.

$$\cos\theta = \frac{v_x}{v_p} = \frac{\frac{\sqrt{2}+1}{1+2\sqrt{2}}c}{c} = \frac{\sqrt{2}+1}{1+2\sqrt{2}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{\sqrt{2}+1}{1+2\sqrt{2}}\right) = 51°$$

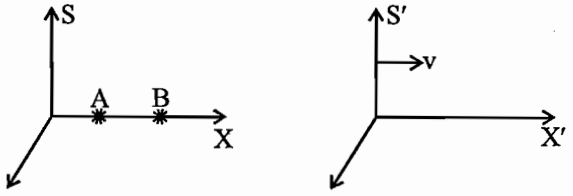
CAREER ENDEAVOUR

Consequences of Relativity:

In Newtonian Mechanics length of an object, time and mass are considered as absolute quantities i.e. their values are independent of frame of reference. However, as it is evident from lorentz transformation, time measurement depends both on state of the observer and its location. Thus, in relativistic mechanics time is not an absolute quantity. In the following we use L-T to demonstrate that length of an object and mass are also not absolute we also discuss other consequences of relativity.

Loss of Simultaneity:

Let two events (say light flashes) take place simultaneously at points $A(x_A, 0, 0)$ and $B(x_B, 0, 0)$ at time $t_A = t_B = t$ in S frame. Therefore time of occurrence of the two events as measured in S′ frame will be



$$t'_A = (t_A - x_A \frac{v}{c^2})\gamma, \quad t'_B = (t_B - x_B \frac{v}{c^2})\gamma$$

Although $t_A = t_B, t'_A \neq t'_B$ because $x_A \neq x_B$.

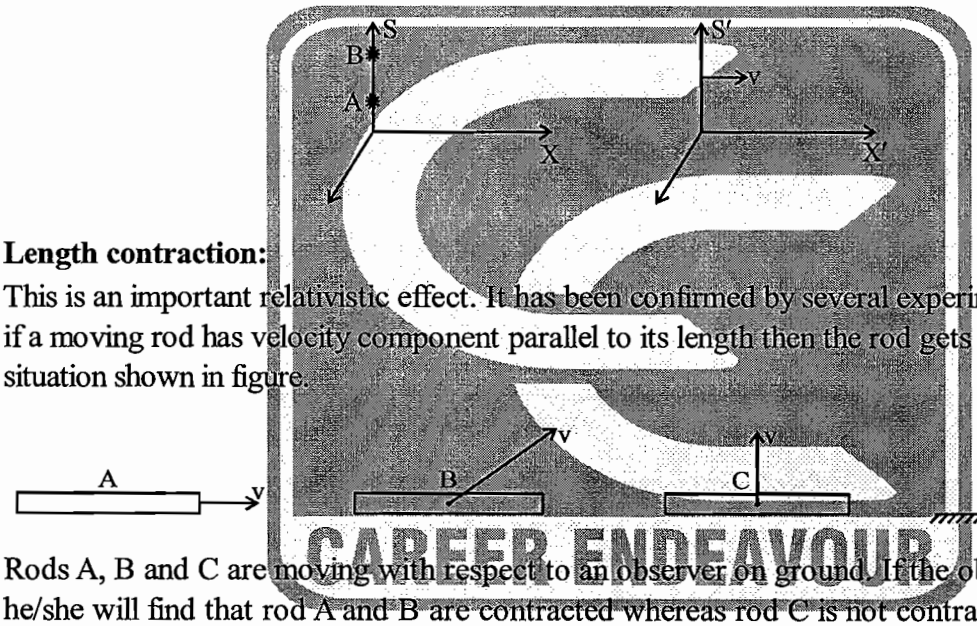
Thus, two events which are simultaneous in S frame will not be simultaneous in S′ frame if S′ has velocity component parallel to line joining the position of two events.

If S′ doesn't have velocity component parallel to the line joining the position of two events then events which are simultaneous in S will also appear simultaneous in S′, as discussed below.

Let two events take place simultaneously at points $A(0, y_A, 0)$ and $B(0, y_B, 0)$ at time $t_A = t_B = t$. Therefore as seen from S′ which is moving along x-direction w.r.t. S, time of occurrence of two events are.

$t'_A = (t_A - x_A \frac{v}{c^2})\gamma \quad \because x_A = 0$ therefore $t'_A = t_A\gamma$

Similarly, $t'_B = t_B\gamma$. And $t_A = t_B$, therefore $t'_A = t'_B$



Length contraction:

This is an important relativistic effect. It has been confirmed by several experiments. This effect tells that if a moving rod has velocity component parallel to its length then the rod gets contracted. Consider three situation shown in figure.

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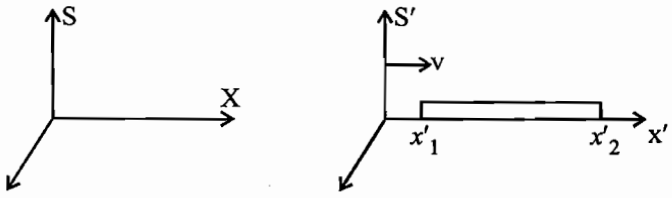
Rods A, B and C are moving with respect to an observer on ground. If the observer does measurement he/she will find that rod A and B are contracted whereas rod C is not contracted.

Let us consider a rod of rest (proper) length L_0 kept in S' frame.

Let x'_1 and x'_2 be coordinates of the ends of rod.

$\therefore x'_2 - x'_1 = L_0$



Rod is at rest in S′ frame so x'_1 and x'_2 need not be measured simultaneously.

To measure length of rod in S frame coordinates of ends must be measured simultaneously because rod is moving w.r.t. S frame.

Let x_1 and x_2 be end coordinates of rod in S frame at time 't'.

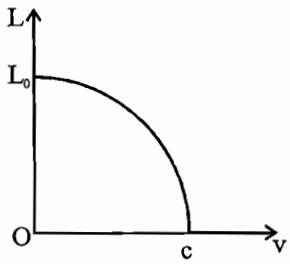
$\therefore\ x_2 - x_1 = L$ (apparent length of rod)

and from L–T

$x_1' = (x_1 - vt)\gamma$ (Note : Here we cannot use inverse L–T)

$x_2' = (x_2 - vt)\gamma$

$\therefore\quad x_2' - x_1' = (x_2 - x_1)\gamma$ or $L_0 = L\gamma$



Variation of L with v

or $$\boxed{L = \frac{L_0}{\gamma} = L_0\sqrt{1-\frac{v^2}{c^2}}} \qquad ...(1.12)$$

$\because\quad \sqrt{1-\frac{v^2}{c^2}} < 1 \quad \therefore\ L < L_0$

Note that in (1.12) v is velocity of the rod relative to observer and it is also velocity parallel to its length.

Example: A rod of rest length L_0 moves with speed $\frac{c}{2}$ with respect to an observer such that velocity of rod makes 45° with its length. Find apparent length of the rod measured by observer.

Soln. Relative velocity of rod parallel to its length is

$$v = \frac{c}{2}\cos 45° = \frac{c}{2\sqrt{2}}$$

$$\therefore\quad L = L_0\sqrt{1-\frac{v^2}{c^2}} = L_0\sqrt{1-\frac{1}{8}} = \frac{\sqrt{7}L_0}{2\sqrt{2}}$$

Example: A rod is moving with speed $\frac{c}{2}$ w.r.t. ground such that velocity of rod is parallel to its length. An observer is also running in the same direction with speed $\frac{c}{3}$ w.r.t. ground. If rest length of the rod is L_0. Find the apparent length of the rod as measured by the observer.

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Soln. Lets call ground to be S frame. Observer is moving so we will call him / her S′ frame. To calculate apparent length we first need velocity of rod w.r.t. to the observer, which is given as (from 1.8).

$$v'_x = \frac{v_x - v}{1-\frac{v_x v}{c^2}}, \qquad \text{here } v_x = \frac{c}{2},\ v = \frac{c}{3} \qquad \therefore\ v'_x = \frac{\frac{c}{2}-\frac{c}{3}}{1-\frac{1}{6}} = \frac{c}{5}$$

Therefore, apparent length of rod is $L = L_0\sqrt{1-\frac{v_x'^2}{c^2}} = L_0\sqrt{1-\frac{1}{25}} = \frac{2\sqrt{6}}{5}L_0$

Relativity of area and volume:

Both area and volume contract in similar manner as length does. If A_0 be proper (rest) area of a surface then its apparent value to an observer w.r.t. whom surface is moving

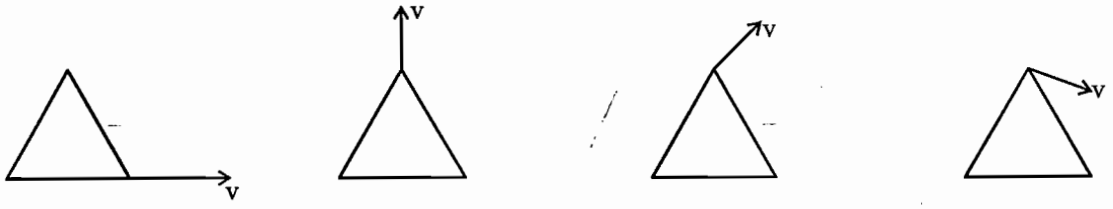
$$A = A_0\sqrt{1-\frac{v^2}{c^2}} \qquad ...(1.13)$$

Here v = relative velocity parallel to the surface.

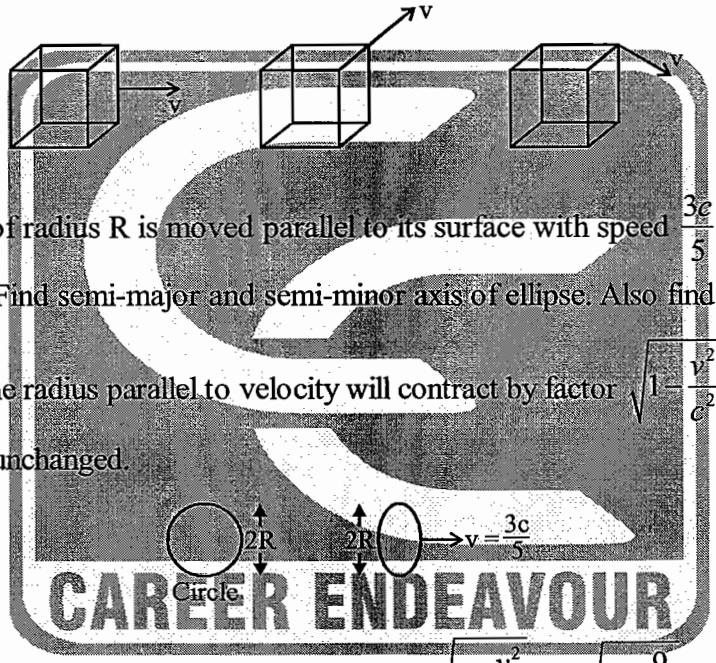
If V_0 be proper volume of an object then its apparent value is

$$V = V_0\sqrt{1-\frac{v^2}{c^2}} \qquad ...(1.14)$$

Here v is net relative velocity (not just a component). In figure shown below, in all cases apparent area of triangle is given by (1.13), therefore value of apparent area is same in all cases.



Similarly, if a cube is translated in any direction, value of apparent volume is same in all cases.



Example: A circle of radius R is moved parallel to its surface with speed $\frac{3c}{5}$ because of which it takes shape of an ellipse. Find semi-major and semi-minor axis of ellipse. Also find apparent area.

Soln. Due to movement the radius parallel to velocity will contract by factor $\sqrt{1-\frac{v^2}{c^2}}$ and that perpendicular to velocity will remain unchanged.

Therefore, Semi-major axis = R, Semi-minor axis = $R\sqrt{1-\frac{v^2}{c^2}} = R\sqrt{1-\frac{9}{25}} = \frac{4R}{5}$

Apparent area $A = A_0\sqrt{1-v^2/c^2} = \pi R^2\sqrt{1-\frac{9}{25}} = \frac{4\pi R^2}{5}$

We can also get above result for apparent area by using formula of ellipse.

Example: An equilateral triangle is moved parallel to its plane in two ways. In first case it is given a speed $\frac{c}{2}$ parallel to one of its sides, and in second case it is given a speed $\frac{\sqrt{3}c}{2}$ parallel to bisector of angle. Find ratio of apparent area in the two cases.

Let A_0 = proper area

In first case, $A_1 = A_0\sqrt{1-\frac{v_1^2}{c^2}} = A_0\sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}A_0}{2}$

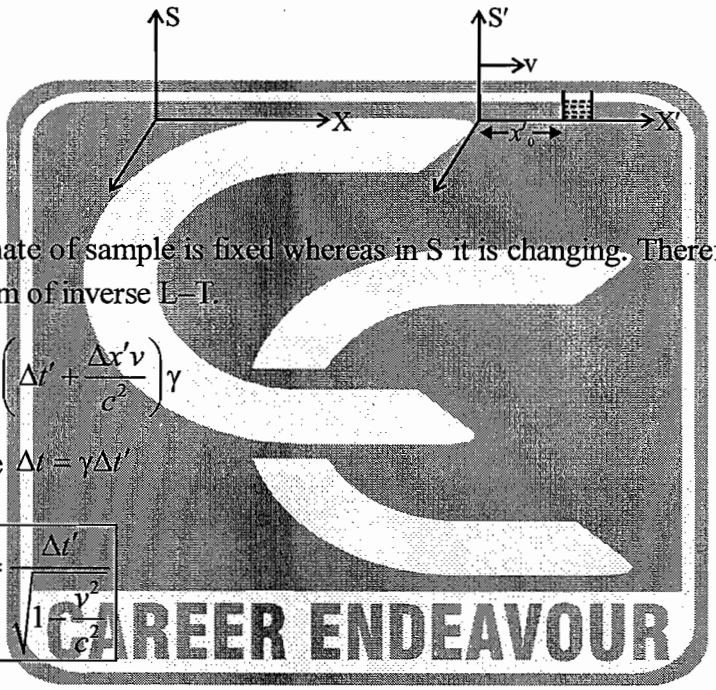
In second case, $A_2 = A_0\sqrt{1-\frac{v_2^2}{c^2}} = A_0\sqrt{1-\frac{3}{4}} = \frac{A_0}{2}$

ratio: $\frac{A_1}{A_2} = \sqrt{3}$

Time dilation:

By time dilation we mean slowing down of a process while in motion w.r.t. an observer due to which completion of process takes more time than in stationary case.

For example, life time of an unstable particular is more when it is in motion than when it is at rest. A moving clock ticks slower than a stationary clock. To see how time interval or time for completion of a process is affected by relative motion, let us consider a freshly prepared sample of radioactive element placed at x_0' in S' frame. Let $\Delta t'$ and Δt be time of decay time measured in S' and S frame respectively.



In S' frame coordinate of sample is fixed whereas in S it is changing. Therefore $\Delta x' = 0$ and $\Delta x \neq 0$. From differential form of inverse L–T.

$$\Delta t = \left(\Delta t' + \frac{\Delta x' v}{c^2}\right)\gamma$$

$\because \Delta x' = 0$, therefore $\Delta t = \gamma \Delta t'$

or $$\boxed{\Delta t = \frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}}} \quad \text{... (1.15)}$$

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We did not want to involve coordinate so we used inverse LT, if we had used direct L-T, Δx would have also come in the expression. $\Delta t'$ is measured in S' frame where $\Delta x' = 0$. Therefore its proper time. It is also called time in rest frame. from (1.15) we find that $\Delta t > \Delta t'$, therefore while in motion radioactive sample appears to decay at slower rate so its completion takes more time.

Example: Life time of a μ meson in its rest frame is 2×10^{-6} sec. What distance can a meson travel before decaying if its speed is 2.994×10^8 m/s.

Soln. Given, $\Delta t' = 2\times10^{-6}$ sec, $v = 2.994\times10^8$ m/s. In lab frame meson is moving so its life time increases. In lab frame life time of meson becomes

$$\Delta t = \frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2\times10^{-6}}{\sqrt{1-\left(\frac{2.994}{3}\right)^2}} = 31.7\times10^{-6}\text{ sec}$$

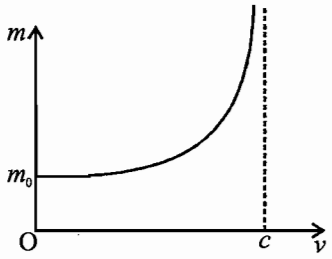
therefore, distance travelled by meson = $v\Delta t = 2.994 \times 10^8 \times 31.7 \times 10^{-6}$ m

$= 9500$ m

Relativity of mass:

Variation of mass with velocity is a profound consequence of relativity. By using conservation of momentum it can be demonstrated that an object whose mass is m_0 when it is at rest, its mass becomes m when its moves with speed v, and 'm' and 'm_0' are related as

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{... (1.16)}$$



Mass energy equivalence:

It tell that greater the energy greater will be mass and vice versa. Thus a compressed or elongated spring which has some potential energy must have slightly greater mass than an identical relaxed spring. By using work energy theorem we can show that an object whose rest mass is m_0 has energy

$$E_0 = m_0c^2 \quad \text{when at rest} \qquad \text{... (1.17)}$$

and $$E = mc^2 = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{when moving with speed } v. \qquad \text{... (1.18)}$$

Relativistic and Non-relativistic quantities:

	Relativistic	*Non-relativistic*
Momentum (p)	mv or $\frac{m_0v}{\sqrt{1-\frac{v^2}{c^2}}}$	m_0v
Kinetic energy (K)	$mc^2 - m_0c^2$ or $\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0c^2$	$\frac{1}{2}m_0v^2$
Total energy of a free particle (E)	mc^2 or $\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}$	$\frac{1}{2}m_0v^2$
Kinetic energy of momentum relation	$pc = \sqrt{K(K+2m_0c^2)}$	$K = \frac{p^2}{2m_0}$
Energy momentum relation	$E = \sqrt{p^2c^2 + m_0^2c^4}$	$E = \frac{p^2}{2m_0}$

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Lorentz invariant quantities:

There are some quantities whose values remains same if they are evaluated in different inertial frames. In the following we give a list of these quantities.

- Rest mass (m_0), proper time ($\Delta t'$), proper length (L_0).
- $E^2 - p^2c^2$, where E = energy, p = momentum.
- $E^2 - B^2c^2$ and $\vec{E}.\vec{B}$, E = electric field, B = magnetic field.

- $x^2+y^2+z^2-c^2t^2$ or $\Delta x^2+\Delta y^2+\Delta z^2-c^2\Delta t^2$ or $\Delta r^2-c^2\Delta t^2$. This quantity is called distance in spacetime four dimensional space.
- Phase of wave, $\vec{k}.\vec{r}-\omega t$
- Maxwell's four equations $\vec{\nabla}.\vec{E}=\frac{\rho}{\varepsilon_0}, \vec{\nabla}.\vec{B}=0, \vec{\nabla}\times\vec{E}=\frac{-\partial\vec{B}}{\partial t}, \vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\mu\in_0\frac{\partial\vec{E}}{\partial t}$
- Equation of wave $\nabla^2\psi=\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$

To show that $x^2+y^2+z^2-c^2t^2$ is Lorentz invariant quantity:

Let's consider two inertial frames S and S' moving with velocity v w.r.t. S along + x-direction. We have to show that,

$$x^2+y^2+z^2-c^2t^2 = x'^2+y'^2+z'^2-c^2t'^2$$

Let's start with R.H.S. and use L–T equation 1.3 and 1.4.

$$x'^2+y'^2+z'^2-c^2t'^2 = \gamma^2(x-vt)^2+y^2+z^2-c^2\gamma^2\left(t-\frac{xv}{c^2}\right)^2$$

$$= \gamma^2\left[x^2+v^2t^2-2xvt-c^2t^2+2xvt-x^2\frac{v^2}{c^2}\right]+y^2+z^2$$

$$= \gamma^2\left[x^2\left(1-\frac{v^2}{c^2}\right)-c^2t^2\left(1-\frac{v^2}{c^2}\right)\right]+y^2+z^2$$

$$= \gamma^2\left(1-\frac{v^2}{c^2}\right)(x^2-c^2t^2)+y^2+z^2$$

$$= x^2+y^2+z^2-c^2t^2, \text{ Q.E.D} \left\{\because \gamma^2\left(1-\frac{v^2}{c^2}\right)=1\right\}$$

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Example: Two events occur on ground at a separation of 5km and at a time difference of 1 ms. The events appear to be occurring at same point in an inertial frame moving along the line joining the position of two events. Calculate the time difference between the two events as measured in the moving frame.

Soln: In ground frame, $\Delta r = 50\text{km}$, $\Delta t = 1\text{ms}$

In moving frame, $\Delta r' = 0$, $\Delta t' = ?$

We know that quantity $\Delta r^2-c^2\Delta t^2$ is invariant under L–T

therefore, $\Delta r^2-c^2\Delta t^2 = \Delta r'^2-c^2\Delta t'^2$

$$25\times10^8 - 9\times10^{16}\times10^{-6} = 0 - 9\times10^{16}\Delta t'^2$$

$$\Delta t' = \sqrt{10^{-6}-\frac{25}{9}\times10^{-8}}, \Delta t' = 0.986\text{ms}$$

Example: Calculate proper time interval between the occurrence of two events if in one inertial frame the events are separated by 3.6×10^8 m and occur 2 sec apart.

Soln: Proper time is the time interval in an inertial frame in which the events occur at same location (i.e. spatial separation is zero)

According to question, $\Delta x = 3.6\times10^8\text{m}$, $\Delta t = 2$ sec in some inertial frame.

Let the time will be proper time in another frame S'.

$\therefore \quad \Delta x' = 0,\ \Delta t'$ = proper time

We know that $\Delta x^2 - c^2\Delta t^2 = \Delta x'^2 - c^2\Delta t'^2$

Therefore, $(3.6 \times 10^8)^2 - 9 \times 10^{16} \times 4 = 0 - 9 \times 10^{16}\Delta t'^2$

or $\Delta t' = \sqrt{4 - \left(\frac{3.6}{9}\right)^2} = 1.6\,\text{sec}$

Energy Momentum Transformation:

Energy of a free particle (in relativistic case) is

$$E = mc^2 = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}, \text{ or } E = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}\frac{\Delta t}{\Delta t}, \quad \because \Delta t\sqrt{1-\frac{v^2}{c^2}} = \Delta t' = \text{proper time}$$

$\therefore \quad E = \frac{m_0c^2}{\Delta t'}.\Delta t$

The quantity $\frac{m_0c^2}{\Delta t'}$ is Lorentz invariant. Therefore under L-T, E should transform in same way as Δt does.

Momentum of a particle (in relativistic case) is

$$p_x = mv_x = \frac{m_0v_x}{\sqrt{1-\frac{v^2}{c^2}}}, \text{ or } p_x = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\cdot\frac{\Delta x}{\Delta t} = \frac{m_0}{\Delta t'}.\Delta x$$

The quantity $\frac{m_0}{\Delta t'}$ is Lorentz invariant. Therefore under L-T, p_x should transform in same way as Δx does.

Thus, to obtain L–T for E and p_x we use L–T for x and t and replaced x by p_x and t by E/c^2 (to be dimensionally consistent).

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Therefore, $x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}$ gives, $\boxed{p_x' = \frac{p_x - \frac{vE}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}}$ ...(1.19)

and $t' = \frac{t - \frac{xv}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$ gives, $\frac{E'}{c^2} = \frac{\frac{E}{c^2} - \frac{p_xv}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$ or $\boxed{E' = \frac{E - p_xv}{\sqrt{1-\frac{v^2}{c^2}}}}$ ...(1.20)

Relativistic Doppler Effect: (Doppler effect of electromagnetic waves):

Change in apparent frequency of an e.m.w. due to relative motion between source and the observer is called relativistic doppler effect.

Expression for apparent frequency can be easily obtained using L-T for energy (equation 1.20).

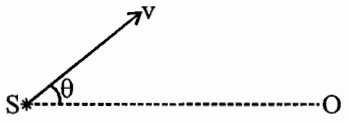
If line joining the source and observer makes an angle θ with the direction of their relative velocity as shown in figure then

$$\nu = \frac{\nu_0\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}\cos\theta} \qquad ...(1.21)$$

where ν = apparent frequency of e.m.w. as seen by observer

ν_0 = Actual frequency of e.m.w. emitted by source

v = relative velocity between source and observer.

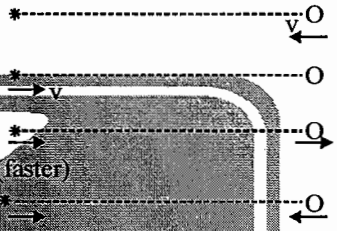


Here, θ is given in the frame of the observer.

Special cases:

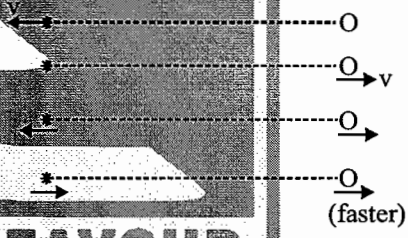
(a) Source and observer moving along the line. Distance between source and observer decreasing. take $\theta = 0$, therefore

$$\nu = \nu_0\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}};\ \nu > \nu_0 \quad \text{or} \quad \lambda = \lambda_0\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}};\ \lambda < \lambda_0$$



(b) Distance between source and observer increasing and the two are moving along the line. take $\theta = 180°$

$$\nu = \nu_0\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\ \nu > \nu_0 \quad \text{or} \quad \lambda = \lambda_0\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}};\ \lambda > \lambda_0$$



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(c) If $\vec{v}$ is perpendicular to the line joining the source and observer at the instant when light is emitted then, $\theta = 90°$.

$$\nu = \nu_0\sqrt{1-\frac{v^2}{c^2}}$$

Example: A light signal is sent from a space-ship towards earth. An observer on earth's space station receives signal in the form of yellow light of wavelength 585 nm. What is the colour of signal sent from spaceship if it is receeding from earth at a constant speed 0.4c

Soln. If λ_0 and λ be the wavelength sent and received respectively, then

$$\lambda = \lambda_0\sqrt{\frac{1+v/c}{1-v/c}}$$

$$585m = \lambda_0\sqrt{\frac{1+0.4}{1-0.4}} \quad \Rightarrow \quad \lambda_0 = 585\sqrt{\frac{3}{14}}\ nm = 270.80\ nm$$

Since, $\lambda_0 < 400nm$ it lies in ultraviolet region of electromagnetic radiation. Therefore signal sent from space-ship is ultraviolet.

Problems on collision or breaking of particles:

In all such problem we use (i) conservation of linear momentum and (ii) conservation of total energy of system.

Therefore we apply,

$\vec{p}_{initial} = \vec{p}_{final}$, $E_{initial} = E_{final}$

Example: A particle of rest mass M initially at rest breaks into two identical particles of rest mass m such that $2m < M$. Calculate momentum of each fragment.

Soln. $2m < M$, there is loss in mass. The lost mass gets converted into kinetic energy of the two particles (conclusion comes from mass energy equivalence). The particle is initially at rest. Therefore $P_{inital} = 0$. Thus, after breaking the two fragments must move in opposite direction with equal magnitude of momentum so that total final momentum remains zero.

Let p be magnitude of momentum of each fragment.

From conservation of total energy $E_{initial} = E_{final}$, $Mc^2 = 2\sqrt{p^2c^2 + m^2c^4}$

$$\frac{M^2c^4}{4} - m^2c^4 = p^2c^2, \text{ or } p = c\sqrt{\frac{M^2}{4} - m^2}$$

Example: Two particles each of rest mass m_0 moving with equal speed $3c/5$ in opposite direction, collide and coalesce to form a new particle. Find mass of new particle.

Soln: Initial momentum is zero, so the new particle will be at rest. Using conservation of total energy we get,

$E_{initial} = E_{final}$

$$\frac{2m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} = Mc^2 \quad \therefore M = \frac{2m_0}{\sqrt{1-\frac{9}{25}}} = \frac{5}{2}m_0$$

CAREER ENDEAVOUR

Example: In a rocket light is used as propellent. Initially rocket is at rest and its rest mass is m_i. It attains some speed by emission of light in opposite direction. If m_f is the final rest mass of rocket, find the speed attained by rocket.

Soln: Let v be the speed of rocket.

Therefore, momentum of rocket = $\dfrac{m_f v}{\sqrt{1-\frac{v^2}{c^2}}}$

Since, initial momentum of rocket was zero, momentum of light emitted must be $\dfrac{m_f v}{\sqrt{1-\frac{v^2}{c^2}}}$ in opposite

direction. Energy emitted in the form of light = $pc = \dfrac{m_f vc}{\sqrt{1-\frac{v^2}{c^2}}}$

From energy conservation $E_{initial} = E_{final}$

$$m_i c^2 = \frac{m_f c^2}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{m_f v c}{\sqrt{1-\frac{v^2}{c^2}}} \text{ or } \frac{m_i}{m_f} = \frac{1+\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \text{ or } \frac{m_i^2}{m_f^2} = \left(\frac{1+\frac{v}{c}}{1-\frac{v}{c}}\right) \therefore v = \left(\frac{m_i^2 - m_f^2}{m_i^2 + m_f^2}\right) c$$

Relation between force and acceleration:

In relativistic mechanics Newton's second law of motion is written as, $\vec{F} = \frac{d\vec{p}}{dt}$

Since mass is not constant we cannot write $F = ma$.

using $\vec{p} = \frac{m_0 \vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}$ we get,

$$\vec{F} = m_0 \frac{d}{dt}\left[\frac{\vec{v}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}\right]$$

Or, $$\vec{F} = \frac{m_0 \vec{a}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} + \frac{m_0 (\vec{v}.\vec{a})\vec{v}/c^2}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} \quad \text{... (1.22)}$$

where, $\vec{a} = \frac{d\vec{v}}{dt}$

Special cases:

(a) If $\vec{v} // \vec{a}$, $\vec{v}.\vec{a} = va$, therefore $F = \frac{m_0 a}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}$ or $\boxed{F = \gamma^3 m_0 a}$

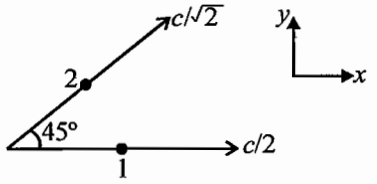
(b) If $\vec{v} \perp \vec{a}$, $\vec{v}.\vec{a} = 0$, therefore $\boxed{F = \gamma m_0 a}$

Example: Two particles are moving with speeds $c/2$ and $c/\sqrt{2}$ at 45° to each other. Calculate their relative velocity.

Soln. **Method 1:** Let $c/2$ is along $+x$ direction and $c/\sqrt{2}$ at 45° with $+x$ direction as show in figure. Let us calculate velocity of 2 w.r.t. 1.

$$\therefore v = \frac{c}{2},\ v_x = \frac{c}{\sqrt{2}}\cos 45° = \frac{c}{2},\quad v_y = \frac{c}{\sqrt{2}}\sin 45° = \frac{c}{2}$$

$$v_x' = \frac{v_x - v}{1-\frac{v_x.v}{c^2}} = \frac{\frac{c}{2}-\frac{c}{2}}{1-\frac{1}{4}} = 0,\quad v_y' = \frac{v_y\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v_x.v}{c^2}} = \frac{\frac{c}{2}\sqrt{1-\frac{1}{4}}}{1-\frac{1}{4}} = \frac{c}{\sqrt{3}}$$



Therefore, relative velocity $= \sqrt{v_x'^2 + v_y'^2} = \frac{c}{\sqrt{3}}$

or, relative velocity vector $\vec{v}' = v_x'\hat{i} + v_y'\hat{j} = \frac{c}{\sqrt{3}}\hat{j}$

Method 2: Velocity of two particles $v_1 = c/2$, $v_2 = c/\sqrt{2}$, $\theta = 45°$
Therefore, relative velocity (using 1.10)

$$v_{21} = \frac{\sqrt{v_1^2 + v_2^2 - 2v_1.v_2\cos\theta - \frac{v_1^2.v_2^2}{c^2}\sin^2\theta}}{1 - \frac{v_1.v_2}{c^2}\cos\theta} = \frac{\sqrt{\frac{c^2}{4} + \frac{c^2}{2} - \frac{c^2}{2} - \frac{c^2}{16}}}{1 - \frac{1}{4}} = \frac{c}{\sqrt{3}}$$

Example: Two event occur in an inertial frame S at a separation of 2×10^6 km and 2 sec apart. In another inertial frame S' which is moving relative to S along the line joining the two events, the two events are simultaneous. Calculate velocity of S' frame.

Soln. In S frame $\Delta x = 2 \times 10^9$m, $\Delta t = 2$sec,
In S' frame $\Delta t' = 0$
using differential form of L-T

$$\Delta t' = \frac{\Delta t - \Delta x.\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}},\quad 0 = \frac{2 - 2\times10^9.\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}},\quad \therefore v = \frac{2\times3\times10^8 c}{2\times10^9},\ v = 0.3c$$

Example: Kinetic energy of a relativistic particle of rest mass m_0 is equal to four times the rest energy of the particle calculate momentum of the particle.

Soln. Rest energy $= m_0c^2$, therefore,

Kinetic energy (K) $= 4m_0c^2$

We know that K.E.(K) and momentum (p) are related as $p = \frac{1}{c}\sqrt{K(K + 2m_0c^2)}$

Therefore, $p = \frac{1}{c}\sqrt{4m_0c^2(4m_0c^2 + 2m_0c^2)} = 4\sqrt{6}m_0c$

Example: Momentum of a particle of rest mass m_0 is m_0c, calculate its energy and speed.

Soln. We know that $E = \sqrt{p^2c^2 + m_0^2c^4}$

Therefore $E = \sqrt{m_0^2c^4 + m_0^2c^4} = \sqrt{2}m_0c^2$

Also, $E = mc^2$ and $p = mv$

$$\therefore \frac{E}{p} = \frac{c^2}{v} \text{ or } v = \frac{pc^2}{E} = \frac{m_0c.c^2}{\sqrt{2}m_0c^2} \quad \therefore v = \frac{c}{\sqrt{2}}$$

Example: Two particles each of rest mass m_0 move with speed v w.r.t. an inertial frame but in opposite direction. Calculate energy of one particle in the rest frame of the other particle.

Soln. Consider two particles A and B as shown in figure. Let us calculate energy of B in $A's$ frame.
In the given frame, energy and momentum of B are

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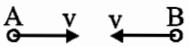
$$E=\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}},\ p_x=-\frac{m_0v}{\sqrt{1-\frac{v^2}{c^2}}}$$



Therefore energy of B in $A's$ frame is

$$E'=\frac{E-p_xv}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}+\frac{m_0v^2}{\sqrt{1-\frac{v^2}{c^2}}}}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{m_0(c^2+v^2)}{1-\frac{v^2}{c^2}}\quad\text{or}\quad E'=m_0c^2.\left(\frac{c^2+v^2}{c^2-v^2}\right)$$

Example: Proper life time of a particle is 5nS. Calculate the distance travelled by the particle before its decay in lab frame where its life time is 10nS.

Soln. **Method 1:** $\Delta t'=5\text{ns},\ \Delta x'=0,\ \Delta t=10\text{ns},\ \Delta x=?$

Using $\Delta x^2-c^2\Delta t^2=\Delta x'^2-c^2\Delta t'^2$

$$\Delta x^2-9\times10^{16}\times10^{-16}=0-9\times10^{16}\times25\times10^{-18}$$

$$\Delta x=3\sqrt{1-0.25}\simeq2.6\text{m}$$

Method 2: $\Delta t=\frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}},\ 10=\frac{5}{\sqrt{1-\frac{v^2}{c^2}}}\Rightarrow v=\frac{\sqrt{3}}{2}c$

Therefore, distance covered $\Delta x=v\Delta t=\frac{\sqrt{3}}{2}\times3\times10^8\times10^{-8}=2.6\text{m}$

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Transformation of force:

Let us consider two inertia frames S and S'. S' moving w.r.t. S along +x direction with speed v.

Force on a particle in S frame,

$$F_x=\frac{dp_x}{dt},\quad F_y=\frac{dp_y}{dt},\quad F_z=\frac{dp_z}{dt}$$

Force on the particle in S' frame

$$F'_y=\frac{dp'_x}{dt'},\quad F'_y=\frac{dp'_y}{dt},\quad F'_z=\frac{dp'_z}{dt}$$

From L-T for momentum we have,

$$dp'_x=\frac{dp_x-v\frac{dE}{c^2}}{\sqrt{1-v^2/c^2}},\quad dp'_y=dp_y,\quad dp'_z=dp_z$$

Therefore,

$$F'_x=\frac{dp'_x}{dt'}=\frac{\frac{dp_x-v/c^2dE}{\sqrt{1-v^2/c^2}}}{\frac{dt-v\frac{dx}{c^2}}{\sqrt{1-v^2/c^2}}}=\frac{\frac{dp_x}{dt}-\frac{v}{c^2}\frac{dE}{dt}}{1-\frac{v}{c^2}\frac{dx}{dt}}=\frac{F_x-\frac{v}{c^2}.\frac{dE}{dt}}{1-\frac{vv_x}{c^2}}$$

$\frac{dE}{dt}$ = power = $\vec{F}.\vec{v}_p$ $\{\vec{v}_p$ = velocity vector of particle)

$$\therefore \quad F'_x = \frac{\vec{F}_x - \frac{v}{c^2}\left(\vec{F}.\vec{v}_p\right)}{1-\frac{v\,v_x}{c^2}}$$

$$\text{Similarly,} \quad F'_y = \frac{F_y\sqrt{1-v^2/c^2}}{1-\frac{v\,v_x}{c^2}}$$

$$F'_z = \frac{F_z\sqrt{1-v^2/c^2}}{1-\frac{v\,v_x}{c^2}}$$

(L-T for force) ... (1.23)

Special case: In the instantaneous rest frame of particle,

$\vec{v}_p = 0$ or $v_x = v_y = v_z = 0$. Therefore,

$$F'_x = F_x$$

$$F'_y = F_y\sqrt{1-v^2/c^2} = \frac{F_y}{\gamma}$$

$$F'_z = F_y\sqrt{1-v^2/c^2} = \frac{F_z}{\gamma}$$

... (1.24)

Classification of interval between two events:

(a) Space like: If $\Delta r > c\Delta t$, spacetime interval $ds^2 = \Delta r^2 - c^2\Delta t^2$ between the two events is called space like because we cannot find an inertial frame in which the two events occure at same spatial point. However, for such events we can find a frame in which the two events occur simultaneously. Therefore this type of events are not causaly connected i.e. one event cannot because for the other event.

(b) Time like: If $\Delta r < c\Delta t$, spacetime interval $ds^2 = \Delta r^2 - c^2\Delta t^2$ between the two events is called time like because we cannot find an inertial frame in which the two events occur simultaneously. However for such events we can find a frame in which the two events occur at space spatial point. Therefore, this type of events are causaly connected. Because of this reason ordering of such events is absolute. The information which connects the two events travels with speed less than speed of light.

(c) Light like: If $\Delta r = c\Delta t$, spacetime interval $ds^2 = \Delta r^2 - c^2\Delta t^2$ between the two events is called light like. In this case information travels with speed of light and two events are causally connected. Therefore, ordering is absolute in this case also.

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ADDITIONAL SOLVED EXAMPLES

Example: A constant force F is applied to a particle of rest mass m_0. The particle is initially at rest. Calculate dynamical mass of the particle after time 't'.

Soln. Rate of change of momentum is equal to force applied.

$$F = \frac{dp}{dt} \quad \Rightarrow \quad p = Ft$$

Therefore, after time 't' energy of the particle becomes

$$E = \sqrt{p^2c^2 + m_0^2c^4} = \sqrt{F^2t^2c^2 + m_0^2c^4}$$

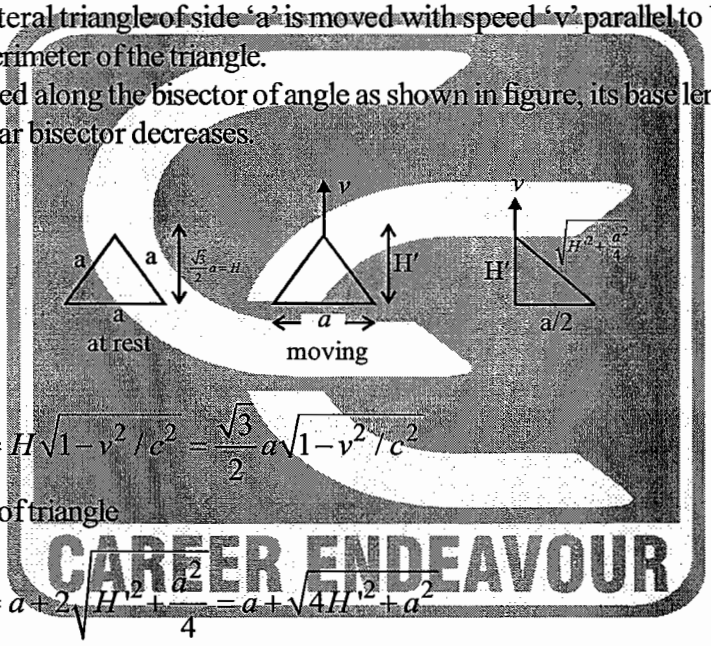
We know that energy is also written as $E = mc^2$, where 'm' is dynamical mass.

$$\therefore \qquad mc^2 = \sqrt{F^2t^2c^2 + m_0^2c^4}$$

$$\text{Or,} \qquad m = \frac{1}{c}\sqrt{F^2t^2 + m_0^2c^2}$$

Example: An equilateral triangle of side 'a' is moved with speed 'v' parallel to bisector of one of its angle. Calculate apparent perimeter of the triangle.

Soln. When triangle is moved along the bisector of angle as shown in figure, its base length remains unchanged but length of perpendicular bisector decreases.



$$\therefore \qquad H' = H\sqrt{1 - v^2/c^2} = \frac{\sqrt{3}}{2}a\sqrt{1 - v^2/c^2}$$

Therefore, perimeter of triangle

$$= a + 2\sqrt{H'^2 + \frac{a^2}{4}} = a + \sqrt{4H'^2 + a^2}$$

$$= a + a\sqrt{3\left(1 - v^2/c^2\right) + 1} = a\left(1 + \sqrt{4 - \frac{3v^2}{c^2}}\right)$$

Example: Half life of an unstable particle is 2μs when it is moving with speed 0.6c. Calculate its half life when it moves with speed 0.8c.

Soln. If t_0 be proper half life, observed half life of a moving particle is given as

$$t = \frac{t_0}{\sqrt{1 - v^2/c^2}}$$

$$\text{Therefore,} \qquad \frac{t_1}{t_2} = \frac{\sqrt{1 - v_2^2/c^2}}{1 - v_1^2/c^2}$$

Here, $t_1 = 2\mu s, v_1 = 0.6c, v_2 = 0.8c, t_2 = ?$

$$\therefore \qquad t_2 = t_1\sqrt{\frac{1 - v_1^2/c^2}{1 - v_2^2/c^2}} = 2\mu s\frac{\sqrt{1 - 0.36}}{\sqrt{1 - 0.64}} = \frac{2 \times 0.8}{0.6}\text{sec} = 2.67 \text{ sec}$$

Example: An electron is accelerated from rest to a speed 0.8c by applying electric potential difference. Calculate the value of potential difference.

Soln. When an electron is accelerated through a potential difference V its kinetic energy becomes, eV. We know that kinetic energy of relativisitic particle is $mc^2 - m_0c^2$. Therefore,

$$\frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2 = eV$$

$$\therefore \quad V = \frac{m_0c^2}{e}\left[\frac{1}{\sqrt{1-v^2/c^2}} - 1\right] = \frac{9.1\times10^{-31}\times9\times10^{16}}{1.6\times10^{-19}} \times \frac{2}{3} = 341 \text{ kilo volt.}$$

Example: A particle of rest mass 'm_1' moving with speed 'v' w.r.t laboratory frame collides with another particle of rest mass m_2 kept initially at rest. After collision the two particles coalasce to form a new entity. Calculate the speed and rest mass of final entity.

Soln.

Let M be the rest mass and v be the speed of final entity. Inorder to conserve momentum the final entity must move in the same direction as v_1 before collision. Using conservation of momentum we get,

$$p_{initial} = p_{final}$$

$$\frac{m_1v_1}{\sqrt{1-v_1^2/c^2}} = \frac{Mv}{\sqrt{1-v^2/c^2}} \quad \text{... (a)}$$

Using conservation of energy we get,

$$E_{initial} = E_{final}$$

$$\frac{m_1c^2}{\sqrt{1-v_1^2/c^2}} + m_2c^2 = \frac{Mc^2}{\sqrt{1-v^2/c^2}} \quad \text{... (b)}$$

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Dividing (b) by (a) we get,

$$\frac{1}{v_1} + \frac{m_2}{m_1v_1}\sqrt{1-v_1^2/c^2} = \frac{1}{v}$$

Or,
$$v = \frac{m_1v_1}{m_1 + m_2\sqrt{1-v_1^2/c^2}}$$

Or,
$$v = \frac{v_1}{1 + \frac{m_2}{m_1}\sqrt{1-v_1^2/c^2}}$$

Put value of 'v' in (a) to get.

$$\frac{m_1v_1}{\sqrt{1-v_1^2/c^2}} = \frac{Mv_1}{\sqrt{\left(1+\frac{m_2}{m_1}\sqrt{1-v_1^2/c^2}\right)^2 - \frac{v_1^2}{c^2}}} \quad \text{... (c)}$$

Let's simplify denominators of R.H.S. =

$$\sqrt{\left(1+\frac{m_2}{m_1}\sqrt{1-\frac{v_1^2}{c^2}}\right)^2-\frac{v_1^2}{c^2}}=\sqrt{1+\frac{m_2^2}{m_1^2}\left(1-v^2/c^2\right)+\frac{2m_2}{m_1}\sqrt{1-v^2/c^2}-\frac{v_1^2}{c^2}}$$

$$=\sqrt{\left(1-\frac{v_1^2}{c^2}\right)\left(1+\frac{m_2^2}{m_1^2}\right)^2+\frac{2m_2}{m_1}\sqrt{1-\frac{v_1^2}{c^2}}}$$

$$=\sqrt{\frac{\left(1-v_1^2/c^2\right)}{m_1^2}\left(m_1^2+m_2^2+\frac{2m_1.m_2}{\sqrt{1-v_1^2/c^2}}\right)}$$

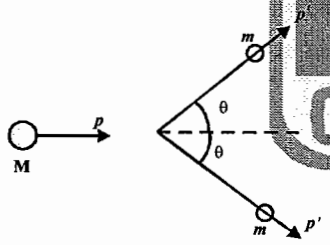
$$=\sqrt{\frac{\left(1-v_1^2/c^2\right)}{m_1}\left(m_1^2+m_2^2+\frac{2m_1.m_2}{1-v_1^2/c^2}\right)}$$

Therefore, using (c) we get

$$M=\sqrt{m_1^2+m_2^2+\frac{2m_1.m_2}{\sqrt{1-v_1^2/c^2}}}$$

Example: A particle of mass 'm' and kinetic energy 'k' collides elastically with an identical particle at rest. After collision the two particles move at equal angle with the incident direction. Calculate the value of angle.

Soln.



Angle is same, so momentum of two particles must be same. Momentum of incident particle is (use p and k relation)

$$p=\frac{1}{c}\sqrt{k\left(k+2m_0c^2\right)} \qquad \text{... (A)}$$

From conservation of momentum,

$$p=2p'\cos\theta$$

$$\cos\theta=\frac{p}{2p'} \qquad \text{... (B)}$$

From conservation of energy,

$$E_{initial}=E_{final}$$

$$k+mc^2+mc^2=2\sqrt{p'^2c^2+m^2c^4}$$

Squaring both sides we get,

$$\left(k+2mc^2\right)^2 = 4\left(p'^2c^2 + m^2c^4\right)$$

$$k^2 + 4kmc^2 = 4p'^2c^2$$

Or, $$p' = \frac{\sqrt{k\left(k+4mc^2\right)}}{2c}$$

From (A) and (B),

$$\cos\theta = \frac{\frac{1}{c}\sqrt{k\left(k+2mc^2\right)}}{2.\frac{\sqrt{k\left(k+4mc^2\right)}}{2c}} = \sqrt{\frac{k+2mc^2}{k+4mc^2}}$$

For angle between the direction of the two particles let us calculate $\cos 2\theta$

$$\cos 2\theta = 2\cos^2\theta - 1 = \frac{2k+4mc^2}{k+4mc^2} - 1 = \frac{k}{k+4mc^2}$$

Therefore, angle between the two particles $= \cos^{-1}\left[\frac{k}{k+4mc^2}\right]$

Example: A particle A of rest mass m_0 has energy $3m_0c^2$. Another identical B moving in same direction as A has energy $\sqrt{2}\,m_0c^2$. Calculate energy of A in rest frame of B.

Soln. For particle A,

$$E_A = 3m_0c^2, \quad P_{x_A} = \frac{1}{c}\sqrt{E_A^2 - m_0^2c^4} = 2\sqrt{2}m_0c$$

For particle B,

$$E'_B = \sqrt{2}\,m_0c^2$$

$$\frac{m_0c^2}{\sqrt{1-v_B^2/c^2}} = \sqrt{2}\,m_0c^2 \qquad \Rightarrow v_B = \frac{c}{\sqrt{2}}$$

Energy of particle A in B's frame

$$E'_A = \frac{E_A - p_{x_A}v_B}{\sqrt{1-v_B{}^2/c^2}} = \frac{3m_0c - 2\sqrt{2}m_0c.\frac{c}{\sqrt{2}}}{\sqrt{1-\frac{1}{2}}} \qquad \Rightarrow E'_A = \sqrt{2}\,m_0c^2$$

Example: A particle of mass 'M' initially at rest breaks up into a particle of mass 'm' and another particle of zero rest mass. Calculate speed of the particle whose rest mass is m.

Soln. From conservation of momentum,

We conclude that momentum of two particles must be equal and opposite.
Let 'v' be the speed of the particle of rest mass 'm'. Speed of the other particle must be equal to speed of light because its rest mass is zero.

For particle 1,

$$p_1 = \frac{mv}{\sqrt{1-v^2/c^2}}; \qquad E_1 = \frac{mc^2}{\sqrt{1-v^2/c^2}}$$

For particle 2,

$$p_2 = p_1 = \frac{mv}{\sqrt{1-v^2/c^2}} \qquad \text{(opposite in direction)}$$

$$E_2 = p_2 c = \frac{mvc}{\sqrt{1-v^2/c^2}}$$

From conservation of eneryg we get,

$$Mc^2 = E_1 + E_2$$

$$Mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} + \frac{mvc}{\sqrt{1-v^2/c^2}}$$

$$\Rightarrow \quad \sqrt{1-v^2/c^2} = \frac{m}{M}\left(1+\frac{v}{c}\right)$$

Or,
$$\sqrt{\frac{1-v/c}{1+v/c}} = \frac{m}{M}$$

$$\therefore \quad v = \frac{M^2-m^2}{M^2+m^2}c$$

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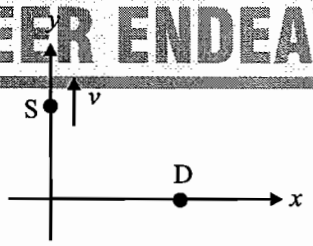
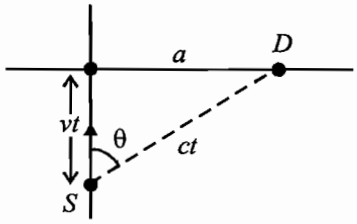
Example: In the figure shown A light source is moving along +Y direction and hight detector is placed at (a, 0, 0). What is the frequency of light measured by the detector at the moment when source was at origin. Actual frequency of light is ν_0.



Soln. The source is moving with speed 'v'. Therefore, light received at any moment must have been emitted by source sometime earlier.

Let light received at the moment when source is at origin was emitted 't' time earlier. Therefore, coordinate of source when it emitted the light is $(0, -vt, 0)$.

$$\therefore \quad \nu' = \frac{\nu_0\sqrt{1-v^2/c^2}}{1-\frac{v}{c}\cos\theta}$$



From figure, $v^2t^2 + a^2 = c^2t^2$

$$t = \frac{a}{c}\cdot\frac{1}{\sqrt{1-v^2/c^2}} \qquad \text{(time at which light was emitted)}$$

And $\cos\theta = \frac{vt}{ct} = \frac{v}{c}$

$\therefore$ $v' = \frac{v_0\sqrt{1-v^2/c^2}}{1-v^2/c^2} = \frac{v_0}{\sqrt{1-v^2/c^2}}$

Example: A rod of proper length L_0 is moving with speed c/2. Parallel to its length. A particle of rest mass m_0 and energy $\frac{2m_0c^2}{\sqrt{3}}$ is moving in opposite direction. Calculate length of rod in rest frame of particle.

Soln. Energy of particle, $E = \frac{2m_0c^2}{\sqrt{3}}$

$\therefore$ $\frac{m_0c^2}{\sqrt{1-v^2/c^2}} = \frac{2m_0c^2}{\sqrt{3}} \Rightarrow v = c/2$

Velocity of rod in given frame $v_x = c/2$

Therefore, velocity of rod in rest frame of particle

$$v'_x = \frac{v_x - v}{1 - \frac{v_x.v}{c^2}} = \frac{c/2-(-c/2)}{1+1/4} = \frac{4c}{5}$$

Apparent length of rod in rest frame of particle,

$$L = L_0\sqrt{1-v'^2_x/c^2} = L_0\sqrt{1-16/25}$$

$$L = \frac{3L_0}{5}$$

CAREER ENDEAVOUR

Lagrangian & Hamiltonian

Classical Mechanics : Its scope and Fundamentals:

Classical Mechanics: Classical or Newtonian mechanics is that branch of physics which deals with motions of bodies travelling at velocities that are much less than that of light in a vacuum. Mechanics of bodies moving at large velocities comparable to that of light is called relativistic mechanics and mechanics that deal with motions of microparticles is called quantum mechanics. Classical mechanics is based on the concepts of absolute mass, distance and time.

The equations of motion in ordinary forms have one disadvantage that they change their form upon coordinate transformation. It would obviously seem to be advatangeous to have the equations of motion in a form which does not suffer from this defect. Moreover, energy of a system is a preferred concept to force. Hence it would be further advantageous to have equations of motion in terms of energy of the system and not in terms of forms. This was done by Joseph Louis Lagrange through the introduction of 'generalised coordinates'. Thereafter many more forms of generalised equations of motions were obtained. These are known as Lagrange's equations of motion, Hamilton's equations of motion, Hamilton-Jacobi equations etc. These equations were obtained from the fundamental principles concerning energy, such as principle of virtual work, Hamilton's principle, principle of least action. These principles are basically the same, that is, they express the same physical fact but their ways of expression are different. Each principle can be derived from the other. The Classical mechanics chapter deals with derivation and application of these generalised equations of motion.

Terms of Principles:

System: A system of particles or bodies (material system) is the conception of a singled-out complex of particles or bodies which, in the general case, interact with one another, as well as bodies not included in the system. A system may consist of only one particle or many particles or bodies. A gas is a system of many particles. The earth-sun system is a material system of two bodies. If their sizes are negligible in comparision to their interdistance, they may be considered as a system of two particles.

Degrees of freedom : Suppose an ant is moving along a straight wire. We require only one coordinate to specify its position at any time. We say that the ant has one degree of freedom. Suppose the same ant is moving on the floor. Now we require two coordinate to specify its position at any time. Here we say that the system has two degrees of freedom. A fly moving in space has three degrees of freedom. Thus, in general, the number of degrees of freedom of a system may be defined as the number of independent variables which are necessary to specify the configuration of any dynamical system. If a system consists of only one particle, it has three degrees of freedom; if it has two particles which are free to be anywhere, then the system has six degrees of freedom and, in general, a system of N particles has $3N$ degrees of freedom.

Constraints and degrees of freedom: More often than not, the motion of the particles of a system is subjected to certain restrictions called 'constraints of motion'. So constraints are some geometrical restrictions to be satisfied by the coordinates or coordinates differences or sometimes by velocities rather than coordinates. For example, in a mole of diatomic gas, there is restriction on the positions of two atoms in a molecule .The restriction is that the distance between the atoms of a single molecule must always be the same. Obviously constraints reduce the number of independent variables required for a system. Here for each molecule we need 5 independent variables instead of six and for a mole of the gas we need 5N independent variables instead of 6N variables. In general if there are k constraints on the system, the number of independent variables needed for a system of N particles is reduced to 3N–k from 3N.

Therefore, degrees of freedom of a system = $3N–k$ where k is the number of restrictions or constraints.

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Types of constraints: If coordinates and velocities of particles of a system are so restricted that they satisfy the equation of the form

$$f(r's, \dot{r}'s, t) = 0$$

that is, some function of coordinates and velocities and time is zero, then this equation is called the constraint equation.

Constraints are called stationary (or scleronomous) if the corresponding constraint equations do not explicitly contain time, otherwise, they are called time-dependent (or rheonomous). Constraints are called holonomic if their equations (that is, constraint equations) do not contain velocities (that is, time derivatives of position coordinates) of particles of the system, that is, restrictions are imposed only on the positions of the particles of the system and not on velocities. However, if restrictions are imposed on velocities and not on coordinates then also constraints may be holonomic if constraints equations can be reduced to a form containing coordinates by integration, that is, the constraint equtaions in terms of velocities are integrable to an equation containing coordinates.

Constraints which cannot be expressed by an equation containing position vectors and time only are called non-holonomic constraints. If there is restrictions on velocity only, the motion is non-holonomic and if restrictions are on both position and velocity, then also motion is non-holonomic.

EXAMPLES:

1. A simple pendulum is a constrained system. It is a holonomic and scleronomous system. The constraint is holonomic because the restriction is imposed on the position of the bob. The distance of the bob from the point of suspension is fixed at the length of the string.
2. The motion of a particle sliding down a plane is holonomic because again here restriction is imposed on the position of the particle, that is, it must always lie on the plane.
3. The motion of a particle sliding down a sphere is nonholonomic because there is only a partial restriction on the position of the particle. So long it is on the surface of the sphere its position is restricted on to it and the motion is holonomic but when it leaves the surface its position may be anywhere outside the surface and so it is non-holonomic.
4. A bead moving on a circular wire is holonomic because position is confined on the wire.
5. The molecules moving inside a gas container is non-holonomic because restrictions on the positions of the molecules are only partial because though they are confined within the vessel, within the vessel there is no restriction on their position.
6. a sphere rolling down a rough inclined plane without slipping is non-holonomic because there restriction is not only on the position but also on the velocity of the point of contact. The condition is that the velocity of the point of contact is equal to zero, that is, $v = \omega r$ where v is the velocity of the centre of mass and ω is the angular velocity of the sphere and r is its radius.

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Generalised coordinates : Generalised velocity : Generalised momentum.

Generalised coordinates : Though Cartesian coordinates are suitable in many problems on account of their utmost simplicity but not always. For example, the motion of a particle under the action of a central force-field is most conveniently solved in polar coordinates (r, θ), the motion of a particle in a spherically symmerical force-field is best solved in spherical polar coordinates (r, θ, φ) etc. So for generality sake it is advantageous and necessary also to have the coordinate system alternative to the Cartesian ones so that they may be particularised wherever necessary. These are called generalised coordinates. Therefore, we may define generalised coordinates as the coordinates thought of in a general way as an alternative to the Cartesian ones. These are denoted by $q_1, q_2, q_3, q_4, \ldots\ldots q_f$ where f is the number of degrees of freedom. These coordinates may be lengths or angles. They are all independent of each other. In a holonomic system (that is, system having restrictions on the positions of its partiles) the absolute values of the position vectors of its particles are connected by the equation of constraints and so the

position vector of any particle is a function of the generalised coordinates of all the particles including that particle and time. Therefore, we have in general in a holonomic system,

$$\vec{r}_i = \vec{r}_i\left(q_1, q_2, q_3, q_4, \ldots\ldots q_f\right)$$

where $\vec{r}_i$ is the position vector of the ith particle. This is called transformation equation. Obviously the equation contains the constraints explicitly.

Generalised velocities: The total time derivatives of the generalised coordinates are called generalised velocities. A system having f degrees of freedom will have f component velocities corresponiding to the f generalised coordinates. These are denoted by

$$\dot{q}_1, \dot{q}_2, \dot{q}_3, \dot{q}_4, \ldots\ldots \dot{q}_f$$

In general $\dot{q}_k = \dfrac{dq_k}{dt}$ where $k = 1, 2, 3, \ldots\ldots f$.

Generalised momenta : The partial derivatives of the Lagrangian with respect to generalised velocities are called generalised momenta. They are denoted by $p_1, p_2, p_3, \ldots\ldots p_f$ where f is the number of degrees of freedom of the system.

In general, $p_k = \dfrac{\partial L}{\partial \dot{q}_k}$, where $k = 1, 2, 3, \ldots\ldots f$.

$$p_k = \frac{\partial T}{\partial \dot{q}_k}$$

$$L = T - V$$

If potential energy depends only on q's and not on $\dot{q}(s)$.

Fundamental Principles : Virtual work : D' Alembert's Principle : Hamilton's Principle : Principle of Least Action.

Virtual displacement : Virtual displacement of a material system are any infinitesimal changes in the configuration of the system, complying with the constraints imposed on the system at a given instant of time. The virtual displacement is denoted by $\delta\vec{r}$. The virtual displacement of the ith particle on the system having f degrees of freedom is $\delta\vec{r}_i$. For a holonomic system, that is, system having constraints on positions of the particles of the system, the position vector of a particle is dependent on the generalised coordinates of the sytem.

$$\therefore \qquad \vec{r}_i = \vec{r}_i\left(q_1, q_2, q_3, \ldots\ldots q_f, t\right)$$

Since virtual displacement is considered at a particular instant, the time dependence of r_i is to be dropped in the consideration of δr_i.

$$\therefore \qquad \delta\vec{r}_i = \sum_{k=1}^{k=f} \frac{\partial\vec{r}_i}{\partial q_k}\partial q_k$$

Virtual work : An element of work done by a force in a virtual displacement is called the virtual work done by that force. If $\vec{F}_i$ is the resultant of the applied force from outside and the force of constraints acting of the ith particle, then the virtual work done by forces on the ith particle, is $\vec{F}_i \cdot \delta\vec{r}_i$ and the total virtual work done on the entire system is $\Sigma\vec{F}_i \cdot \delta\vec{r}_i$ where the summation runs over all the n particles of the system.

Ideal Constraints : Constraints are said to be ideal (workless) if the sum of virtual work done by constraining forces on the particles of the system is zero. That is, for a system of ideal constraints,

$$\sum_{i=1}^{i=n} \vec{F}_i^c \cdot \delta\vec{r}_i = 0$$

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where 'i' stands for ith particle and 'c' for constraint. A particle in circular motion has ideal constraints. A sphere rolling down a plane without slipping has ideal constraints.

Principle of Virtual work: This principle states that it is necessary and sufficient for the equilibrium of any material system with ideal constraints that the sum of elements of work, performed by the applied forces acting on the system in any virtual displacement must be equal to zero. Mathematically, therefore, it stastes that,

$$\sum_{i=1}^{i=n} \vec{F}_i^a \cdot \delta \vec{r}_i = 0$$

here 'i' stands for ith particle and 'a' for applied forces.

$$\boxed{\sum_{i=1}^{i=n} \left(\vec{F}_i^a - \dot{\vec{p}}_i\right) \cdot \delta \vec{r}_i = 0}$$ D'Alembert's princple.

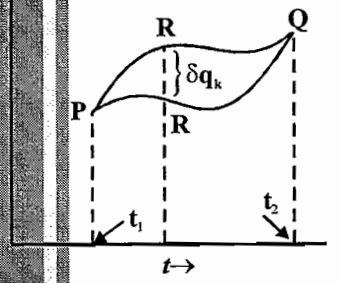
We will refer to this equivalent form as D'Alembert's principle.

Hamilton's Principle : Hamilton's principle states that out of the infinite kinematically possible paths between two states of a holonomic conservative system the actual course of the system is that one along which the time integral of the difference of the kinetic energy and potential energy is stationary, that is, variation of the integral is zero.

Analytically, $\int_{t_1}^{t_2} (T - V)\, dt$ = extremum (or stationary

Or, $\delta \int_{t_1}^{t_2} (T - V)\, dt = 0$

Or, $\int_{t_1}^{t_2} \delta (T - V)\, dt = 0$



An illustration of δ-variation

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Where δ stands for a special type of variation between two coterminus paths. Setting the first variation of the action to zero gives the ***Euler-Lagrange equations,***

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

momentum p_i over $\frac{\partial L}{\partial \dot{q}_i}$; Force F_i over $\frac{\partial L}{\partial q_i}$

Thus, we have the familiar, $\dot{p}_i = F_i$, also known as Newton's second law. Note, however, that the $\{q_i\}$ are generalized coordinates, so p_i may not have dimensions of momentum, nor F_i of force. For example, if the generalized coordinate in question is an angle ϕ, then the corresponding generalized momentum is the angular momentum about the axis of ϕ's rotation, and the generalized force is the torque.

Invariance of the equations of motion:

The equations of motion are invariant under a shift of L by a total time derivative of a function of coordinates and time i.e.

$$\bar{L}(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} G(q, t)$$

Theorem of quadratic nature of kinetic energy :

The kinetic energy of a system is, in general, a quadratic function of generalised velocities and, in particular, a homogeneous quadratic function of generalised velocites and, in particular, a homogeneous quadratic function of generalised velocities when the kinetic energy is independent of time.

Proof : From the fundamental formula we have

$$T=\sum_{i=1}^{i=n}\frac{1}{2}m_i\dot{r}_i^2=\sum_{i=1}^{i=n}\frac{1}{2}m_i\vec{\dot{r}}_i\cdot\vec{\dot{r}}_i \quad \left(\because \vec{\dot{r}}_i\cdot\vec{\dot{r}}_i=\dot{r}_i^2\right)$$

where n is the number of particles of the system, m_i is the mass of the ith particle and $\dot{r}_i$ is the velocity of the particle.

Now, $\vec{r}_i=\vec{r}_i\left(q_1,q_2,.....,q_f,t\right)$ where f is the number of degrees of freedom.

$$\therefore \quad \vec{\dot{r}}_i=\frac{d\vec{r}_i}{dt}=\sum_{k=j}^{k=f}\frac{\partial\vec{r}_i}{\partial q_k}\frac{dq_k}{dt}+\frac{\partial\vec{r}_i}{\partial t}=\sum_{k=1}^{k=f}\frac{\partial\vec{r}_i}{\partial q_k}\dot{q}_k+\frac{\partial\vec{r}_i}{\partial t}$$

$$\therefore \quad T=\sum_{i=1}^{i=n}\frac{1}{2}m_i\left(\sum_{k=1}^{k=f}\frac{\partial\vec{r}_i}{\partial q_k}\dot{q}_k+\frac{\partial\vec{r}_i}{\partial t}\right)\cdot\left(\sum_{j=1}^{j=f}\frac{\partial\vec{r}_i}{\partial q_j}+\frac{\partial\vec{r}_i}{\partial t}\right)$$

Or, $$T=\sum_{i=1}^{i=n}\frac{1}{2}m_i\sum_{k=1}^{k=f}\sum_{j=1}^{j=f}\left(\frac{\partial\vec{r}_i}{\partial q_k}\cdot\frac{\partial\vec{r}_i}{\partial q_j}\dot{q}_k\dot{q}_j+\frac{\partial\vec{r}_i}{\partial q_k}\cdot\frac{\partial\vec{r}_i}{\partial t}\dot{q}_k+\frac{\partial\vec{r}_i}{\partial t}\cdot\frac{\partial\vec{r}_i}{\partial q_j}\dot{q}_j+\frac{\partial\vec{r}_i}{\partial t}\cdot\frac{\partial\vec{r}_i}{\partial t}\right)$$

Since the second and third terms are identical, they may be added.

$$\therefore \quad T=\frac{1}{2}\sum_{i=1}^{i=n}m_i\sum_{k=1}^{k=f}\sum_{j=1}^{j=f}\frac{\partial\vec{r}_i}{\partial q_k}\cdot\frac{\partial\vec{r}_i}{\partial q_j}\dot{q}_k\dot{q}_j+\frac{1}{2}\sum_{i=1}^{i=n}m_i\sum_{k=1}^{k=f}2\frac{\partial\vec{r}_i}{\partial q_k}\dot{q}_k+\sum_{i=1}^{i=n}\frac{1}{2}m_i\left(\frac{\partial\vec{r}_i}{\partial t}\right)^2$$

$$=\sum_{k=1}^{k=f}\sum_{j=1}^{j=f}a_{kj}\dot{q}_k\dot{q}_j+\sum_{k=1}^{k=f}a_k\dot{q}_k+a$$

Where, $a_{kj}=\sum_{i=1}^{i=n}\frac{1}{2}m_i\frac{\partial\vec{r}_i}{\partial q_k}\cdot\frac{\partial\vec{r}_i}{\partial q_j}$ $a_k=\sum_{i=1}^{i=n}m_i\frac{\partial\vec{r}_i}{\partial q_k}\cdot\frac{\partial\vec{r}_i}{\partial t}$

$$\therefore \quad a=\sum_{i=1}^{i=n}\frac{1}{2}m_i\left(\frac{\partial\vec{r}_i}{\partial t}\right)^2$$

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From this equation it is seen that the kinetic energy is, in general, a quadratic function of generalised velocities. A case of of considerable importance ariese when t is not explicitly involved. Then $\frac{\partial\vec{r}_i}{\partial t}=0$ and , therefore, $a_k=0$ and $a=0$, so the above equation reduces to

$$T=\sum_{k=1}^{k=f}\sum_{j=1}^{j=f}a_{jk}\dot{a}_k\dot{q}_j$$

Thus the kinetic energy is a homogeneous quadratic expression of generalised velocitie when it is independent of time. In a free system, conservative system or system with scleronomous constraints the kinetic energies are independent of time and so their kinetic energies are homogeneous quadratic function of generalised velocities.

Cyclic or Ignorable coordinates : Principles of Homogenity of time and space : Deduction of conservation theorems.

Cyclic or ignorable coordinates : If the Lagrangian of a system is not the function of a given coordinate, then that coordinate is said to be cyclic or ignorable coordinate of the system. If the coordinate q_k does not occur in the expression for L, then q_k is said to be ignorable or cyclic.

In such case $$\frac{\partial L}{\partial q_k} = 0$$

(a) Conservation theorem for generalised momentum : The theorem is : the generalised momentum corresponding to a cyclic or ignorable coordinate of conservative system is conserved.

Proof : According to Lagrange's equations for a conservative system,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$$

If the k-th generalised coordinate is ignorable, then $\frac{\partial L}{\partial q_k} = 0$

With this substitution in the Lagrange's equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0 \quad \text{or} \quad \frac{\partial L}{\partial \dot{q}_k} = a \text{ constant.}$$

But $\frac{\partial L}{\partial \dot{q}_k} = p_k$, k-th generalised momentum of the sytem.

Therefore, p_k = constant

(b) Conservation theorem for energy : The principle of homogeneity of time : The theorem states : If the Lagrangian of a system does not contain time explicitly the total energy of the system remains conserved. This follows from the principle of homogeneity of time which states that the laws of motion of a system does not depend on the choice of time reference.

Proof : If lagrangian does not contain time, then it is a function of generalised coordinates and velocities only

That is, $$L = L(q, \dot{q})$$

The total time derivative of it is given by

$$\frac{dL}{dt} = \sum_{k=1}^{k=f}\left(\frac{\partial L}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt}\right)$$

where, f = number of degrees of freedom.

Using Lagrange's equations $\frac{d}{dt}\left(\frac{dL}{d\dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$ we have after replacing $\frac{\partial L}{\partial q_k}$ in the summation by $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)$

$$\frac{dL}{dt} = \sum_{k=1}^{k=f}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\frac{dq_k}{dt} + \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt}\right] = \sum_{k=1}^{k=f}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k\right]$$

Using the rule for differentiation of the product $\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k$ we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k$$

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$$\therefore \quad \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k$$

$$\therefore \quad \frac{dL}{dt} = \sum_{k=1}^{k=f}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k\right] = \sum_{k=1}^{k=f}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right)$$

Interchanging summation with the total time differential,

$$\frac{dL}{dt} = \frac{d}{dt}\sum_{k=1}^{k=f}\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k$$

Or, $$\frac{d}{dt}\left[L - \sum_{k=1}^{k=f}\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right] = 0$$

Or, $$L - \sum_{k=1}^{k=f}\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k = a \text{ constant}$$

But, $$\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}(T - V) = \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial V}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k}$$

(Since V is a function of q's only)

T is a homogeneous second degree function of generalised velocities, that is, $T = \left(a_1\dot{q}_1^2 + a_2\dot{q}_2^2 + \ldots\ldots + \dot{q}_f^2 a_f + b_1\dot{q}\dot{q}_2 + \ldots\ldots\right)$.

Where a's and b's are constants depending on masses of particles and generalised coordinates.

By Euler's theorem,

$$\sum_{k=1}^{k=f}\dot{q}_k\frac{\partial T}{\partial \dot{q}_k} = 2T \text{ or } \sum_{k=1}^{k=f}\dot{q}_k\frac{\partial L}{\partial \dot{q}_k} = 2T$$

$\therefore$ $L - 2T = a$ constant or $(T - V) - 2T = a$ constant

Or, $T + V = a$ constant

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(c) Conservation of linear momentum : The principle of homogeneity of space. The theorem states that the linear momentum of a closed system (that is, a system free form external forces) is a constant (called an integral of motion) of the system.

Proof : The law follows from the **principle of homogeneity of space** which means that a parallel translation of a closed system as a whole in no way changes the mechanical property of the system i.e. its Lagrangian function is left unchanged. In other words we may say that the principle of homogeneity of space means that mechanical property of a closed system cannot change if origin of the frame of reference is chanaged.

The Lagrangian function of a closed system consisting of n particles is given by

$$L = T - V = \sum_{i=1}^{i=n}\frac{1}{2}m_i\dot{r}_i^2 - V\left(\vec{r}_1 \cdot \vec{r}_2 \ldots\ldots \vec{r}_n\right)$$

The change in the Lagrangian function upon a parallel translation of the system through an infinitesimal distance, specified by an arbitrary value δs, is

$$\partial L = \sum_{i=1}^{i=n}\frac{\partial L}{\partial r_i}\delta s = \delta s\sum_{i=1}^{i=n}\frac{\partial L}{\partial r_i}$$

According to the principle of homogeneity of space this change of the Lagrangian function must be zero. Hence, for compliance with this fundamental principle we have

$$\sum_{i=1}^{i=n} \frac{\partial L}{\partial r_i} = 0$$

According to Lagrangian's equation of motion, equation for the linear motion of the i-th particle is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) - \frac{\partial L}{\partial r_i} = Q'_i \text{ or } \sum_{i=1}^{i=n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) - \sum_{i=1}^{i=n} \frac{\partial L}{\partial r_i} = \sum_{i=1}^{i=n} Q'_i$$

Where Q'_i is the resultant of all non-conservative forces acting on the ith particle of the system. In case of a closed system, these forces are only internal in origin as a closed system means a system free from external forces, that is, for such a system $\sum_{i=1}^{i=n} Q'_i = 0$ because internal forces being equal and opposite their sum is zero.

$$\therefore \quad \sum_{i=1}^{i=n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) = 0 \text{ or } \frac{d}{dt}\sum_{i=1}^{i=n} \frac{\partial L}{\partial \dot{r}_i} = 0$$

$$\text{Or,} \quad \sum_{i=1}^{i=n} \frac{\partial L}{\partial \dot{r}_i} = a \text{ constant, or } \sum_{i=1}^{i=n} m_i v_i = a \text{ constant}$$

$$\left(\because \frac{\partial L}{\partial \dot{r}_i} = \text{linear momentum of i-th particle}\right)$$

How to solve Mechanics problems:

Here are some simple steps you can follow toward obtaining the equations of motion.

(1) Choose a set of generalized coordinates $\{q_i \ldots\ldots, q_n\}$

(2) Find the kinetic energy $T(q,\dot{q},t)$, the potential energy $U(q,t)$, and the Lagrangian $L(q,\dot{q},t) = T - U$. It is often helpful to first write the kinetic energy in Cartesian coordinates for each particle before converting to generalized coordinates.

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(3) Find the canonical momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$ and the generalized forces $F_i = \frac{\partial L}{\partial q_i}$

(4) Evaluate the time derivaties $\dot{p}_i$ and write the equations of motion $\dot{p}_i = F_i$. Be careful to differentiate properly, using the chain rule and the Leibniz rule where appropriate.

(5) Identify any conserved quantities (more about this later).

EXAMPLES

1. Set up the Lagrangian function of a simple pendulum and hence obtain the equation describing its motion.

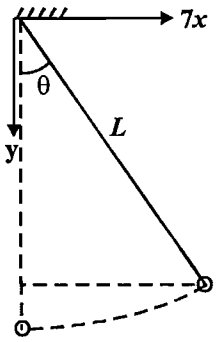
Soln. Let us consider the horizontal line to the right through the point of suspension of as x-axis and the vertical downward as y-axis. Let r, θ be the polar coordinate of the bob of the pendulum at any instant. The constraint on the motion of the pendulum is that

r (modulus of the position vector) = l, length of the string. Since the gravitational field-force is conservative, a simple pendulum is a holonomic conservative system.

Here, $q_1 = r = \ell$ (fixed) and $q_2 = \theta$

$$\text{T (kinetic energy)} = \frac{1}{2}\ell\dot{\theta}^2 = \frac{1}{2}m\dot{r}^2\dot{\theta}^2 = \frac{1}{2}m\ell^2\dot{\theta}^2$$

$$\text{V (potential energy)} = -mg\ell\cos\theta$$



Therefore, $T - V = \frac{1}{2}m\ell^2\,\dot{\theta}^2 + mg\ell\cos\theta$

$\therefore$ $\frac{\partial L}{\partial\dot{\theta}} = m\ell^2\,\dot{\theta}$ and $\frac{\partial L}{\partial\theta} = -mg\ell\sin\theta$

$\because$ $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta}$

$\therefore$ $\frac{d}{dt}\left(m\ell^2\dot{\theta}\right) = -mg\ell\sin\theta$

Or, $\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$

This is the required equation describing the motion of a simple pendulum.

2. A particle of mass m is projected with an initial velocity u at an angle α with the horizontal. Use Lagrange's equations to describe the motion of projectile. The resistance of air is negligible.

Soln. Let x, y be the coordinate of the particle at any instant. The system is holonomic because its position is confined to a plane and conservative because the only external force acting on the projectile is conservative.

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \text{ and } V = mgy$$

$\therefore$ $L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$

$$\frac{\partial L}{\partial\dot{x}} = m\dot{x} \text{ and } \frac{\partial L}{\partial x} = 0$$

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By Lagrange's equation, $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) = \frac{\partial L}{\partial x}$

$\therefore$ $\frac{d}{dt}(m\dot{x}) = 0$ or $\ddot{x} = 0$

$$\frac{\partial L}{\partial\dot{y}} = my \text{ and } \frac{\partial L}{\partial y} = -mg$$

By Lagrange's equation $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{y}}\right) = \frac{\partial L}{\partial y}$

$\therefore$ $\frac{d}{dt}(my) = -mg$ or $\ddot{y} = -g$

Thus, $\ddot{x} = 0$, $\ddot{y} = -g$ are the two equations describing the motion of a projectile.
Proceeding with the first equation,

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = 0 \text{ or } d\left(\frac{dx}{dt}\right) = 0$$

Integrating, $\frac{dx}{dt} = c_1$ (a constant)

At $t = 0$, $\frac{dx}{dt} = u\cos\alpha$; $\therefore u\cos\alpha = c_1$

$\therefore$ $\frac{dx}{dt} = u\cos\alpha$, or $dx = u\cos\alpha\, dt$

Integrating, $x = (u\cos\alpha)t + x_0$ (a constant)

At $t = 0, x = 0$, therefore, $x_0 = 0$; $x = (u\cos\alpha)t$

Proceeding with the second equation,

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = -g, \text{ or } d\left(\frac{dy}{dt}\right) = -g\,dt$$

Integrating, $\frac{dy}{dt} = -gt + c_2$ (a constant)

At $t = 0$, $\frac{dy}{dt} = u\sin\alpha$; $\therefore u\sin\alpha = c_2$

$\therefore$ $\frac{dy}{dt} = -gt + u\sin\alpha$, or $dy = -gt\,dt + u\sin\alpha\, dt$

Integrating, $y = -\frac{1}{2}gt^2 + (u\sin\alpha)t + y_0$

At $t = 0, y = 0$ and therefore, $y_0 = 0$

$\therefore$ $y = (u\sin\alpha)t - \frac{1}{2}gt^2$

Thus, $x = (u\cos\alpha)t$, $y = (u\sin\alpha)t - \frac{1}{2}gt^2$ are the two solved equation describing the motion of a projectile.

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3. Use Lagrange's method to find the common acceleration of two bodies connected by an inextensible string passing over a frictionless light pulley.

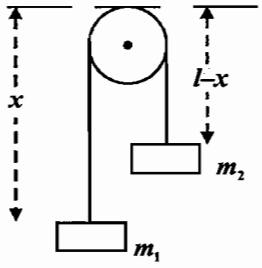
Soln. Let l be the length of the string. The system is conservative and holonomic because the gravitational field-force is conservative and the constraint is on the positions of the masses. Let us take the horizontal line through the pulley as the reference line and the topmost point of the pulley as origin. If x be the coordinate of m_1, then the coordinate of the other is $\ell - x$. So it is a system of only one coordinate and, therefore, there will be only one Lagrange's equation describing their motion.

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

$$V = -m_1 gx - m_2 g(\ell - x)$$

$\therefore$ $$L = T - V = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1\dot{g}x + m_2 g(\ell - x)$$

$$\frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x} \text{ and } \frac{\partial L}{\partial x} = m_1 g - m_2 g = (m_1 - m_2)g$$



By Lagrange's equation, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=\frac{\partial L}{\partial x}$

$\therefore \quad \frac{d}{dt}(m_1+m_2)\dot{x}=(m_1-m_2)g$

Or, $(m_1+m_2)\ddot{x}=(m_1-m_2)g$

Or, $\ddot{x}=\frac{m_1-m_2}{m_1+m_2}g$

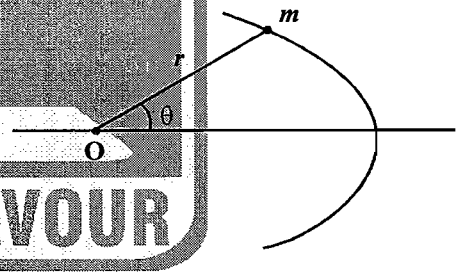
4. A particle of mass m is moving in a plane under the influence of a force directed towards a fixed point and varying inversely as the square of the distance from that point. Set up the Lagrangian and equations of motion of the particle.

Soln. The problem is best solved in polar coordinates with respect to the fixed point as origin and any line through it as the reference line. Let (r,θ) be the polar coordinates of the particle at any time. Since there are two coordinates there will be two Lagrange's equations describing the motion of the particle. Motion is holonomic because the position of the particle is confined to a plane and conservative because the force is derivable from energy or energy is derivable from the force.

$T=\frac{1}{2}m\dot{r}^2$ (linear kinetic energy) $+\frac{1}{2}I\dot{\theta}^2$ (rotational k. energy)

Or, $T=\frac{1}{2}m\dot{r}^2+\frac{1}{2}\left(mr^2\right)\dot{\theta}^2 \qquad \left(\because I=mr^2\right)$

Since force varies inversely as the square of the distance and it is directed towards the origin, $F=-\left(k/r^2\right)$ where k is a constant. But



$F=-\left(\frac{dV}{dr}\right)$

$\therefore \quad =\frac{k}{r^2}=-\frac{dV}{dr}$ or $dV=-\frac{kdr}{r^2}$

Integrating, $V=-\frac{k}{r}+V_0$ (a constant)

At $r=\infty$, $V=0$ and therefore, $V_0=0$; $\therefore V=-\left(\frac{k}{r}\right)$

$\therefore \quad L=T-V=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+\frac{k}{r}$

$\frac{\partial L}{\partial \dot{r}}=m\dot{r}$ and $\frac{\partial L}{\partial r}=mr\dot{\theta}^2-\frac{k}{r^2}$

$\frac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta}$ and $\frac{\partial L}{\partial \theta}=0$

By Lagrange's equations of motion, $\left[\frac{d}{dt}\left(\frac{\partial L}{\partial q_k}\right)=\frac{\partial L}{\partial q_k}\right]$ we have

$\frac{d}{dt}(m\dot{r})=mr\dot{\theta}^2-\frac{k}{r^2}$ and $\frac{d}{dt}\left(mr^2\dot{\theta}\right)=0$

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Or, $\ddot{r} = r\dot{\theta}^2 - \frac{k}{mr^2}$ and $r^2\dot{\theta} = \text{constant}$

Or, $\ddot{r} - r\dot{\theta}^2 = -\frac{k}{mr^2}$ or $\ddot{r} - r\dot{\theta}^2 = -\frac{\omega^2}{r^2}$

where, $\omega^2 = \frac{k}{m}$ and $r^2\dot{\theta} = h$ (another constant).

Thus equations describing the motion of the particles are

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\omega^2}{r^2} \text{ and } r^2\dot{\theta} = h \text{ where } \omega^2 \text{ and } h \text{ are constants.}$$

5. Obtain the Lagrangian of a linear simple harmonic oscillator and the equation of motion in one dimension.

Soln. In a simple harmonic oscillator the particle is acted on by a force which is directed towards a fixed point (position of equilibrium) and whose magnitude varies linearly with the distance from the position of equilibrium. That is, $F = -kx$ where x is the distance of the particle from the equilibrium position.

But $F = -\left(\frac{dV}{dx}\right)$; $\because dV = -Fdx = kxdx$

Integrating, $V = \frac{1}{2}kx^2 + V_0 \,(a \text{ constant})$

At $x = 0, V = 0$ and therefore, $V_0 = 0$

$\therefore$ $V = \frac{1}{2}kx^2$; $T = \frac{1}{2}m\dot{x}^2$

$\therefore$ $L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

Now, $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$ and $\frac{\partial L}{\partial x} = -kx$

By Lagrange's equation, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$

$\therefore$ $\frac{d}{dt}(mx) = -kx$, or $m\ddot{x} = -kx$

This is the equation of a simple harmonic oscillator in one dimension.

6. Obtain the Lagrangian of a charged particle of mass *m* and charge *e*, moving with a velocity *v* in an electric field $\vec{E}$ and magnetic field $\vec{B}$.

Soln. The force experienced by a moving charged particle is given by

$$\vec{F} = e\left(\vec{E} \times \vec{v} \times \vec{B}\right)$$ Lorentz force

It is a fact that in electrodynamics corresponding to the electric and magnetic field there exists a scalar function φ called scalar potential of the field and a vector function $\vec{A}$ called vector potential such that

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} \text{ and } \vec{B} = \vec{\nabla} \times \vec{A}$$

If we substitute these values in the above formula for $\vec{F}$ we have

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$$\vec{F} = e\left\{-\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} + \vec{v}\times\left(\vec{\nabla}\times\vec{A}\right)\right\}$$

We have the identity

$$\vec{v}\times\left(\vec{\nabla}\times\vec{A}\right) = \vec{\nabla}\left(\vec{v}\cdot\vec{A}\right) - \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A}$$

$$\therefore \quad \vec{F} = e\left\{-\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\left(\vec{v}\cdot\vec{A}\right) - \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A}\right\}$$

$$\because \quad \vec{A} = \vec{A}(x, y, z, t)$$

$$\therefore \quad \frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial x}\dot{x} + \frac{\partial \vec{A}}{\partial y}\dot{y} + \frac{\partial \vec{A}}{\partial z}\dot{z} + \frac{\partial \vec{A}}{\partial t}$$

$$= \left(\vec{i}v_x + \vec{j}v_y + \vec{k}v_z\right)\cdot\left(\vec{i}\frac{\partial \vec{A}}{\partial x} + \vec{j}\frac{\partial \vec{A}}{\partial y} + \vec{k}\frac{\partial \vec{A}}{\partial t}\right) + \frac{\partial \vec{A}}{\partial t} \quad \left(\because\ \dot{x} = v_x,\ \dot{y} = v_y \text{ and } \dot{z} = \mathrm{v}_z\right)$$

$$= \left\{\left(\vec{i}v_x + \vec{j}v_y + \vec{k}v_z\right)\cdot\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\right\}\vec{A} + \frac{\partial \vec{A}}{\partial t}$$

$$= \left(\vec{v}.\vec{\nabla}\right)\vec{A} + \frac{\partial \vec{A}}{\partial t}$$

$$\therefore \quad \vec{F} = e\left[-\vec{\nabla}\varphi - \frac{d\vec{A}}{dt} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A} + \vec{\nabla}\left(\vec{v}\cdot\vec{A}\right) - \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A}\right]$$

$$= e\left[-\vec{\nabla}\left(\varphi - v\cdot\vec{A}\right) - \frac{d\vec{A}}{dt}\right]$$

But $\vec{F} = m\frac{d\vec{v}}{dt} = \frac{d}{dt}\left(m\vec{v}\right)$

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$$\therefore \quad \frac{d}{dt}\left(m\vec{v}\right) = e\left[-\vec{\nabla}\left(\varphi - \vec{v}\cdot\vec{A}\right) - \frac{d\vec{A}}{dt}\right]$$

Or, $$\frac{d}{dt}\left(m\vec{v} + e\vec{A}\right) + e\vec{\nabla}\left(\varphi - \vec{v}\cdot\vec{A}\right) = 0$$

This is of the Lagrandian form $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$

Taking $q_k = \vec{r}$ for the particle we have by comparison

$$\frac{\partial L}{\partial \dot{r}} = m\vec{v} + e\vec{A} \text{ and } \frac{\partial L}{\partial \vec{r}} = -e\vec{\nabla}\left(\varphi - \vec{v}\cdot\vec{A}\right)$$

Since, $\vec{\dot{r}} = \vec{v}$

$$\therefore \quad \frac{\partial L}{\partial \vec{\dot{r}}} = m\vec{\dot{r}} + e\vec{A} \text{ and } \frac{\partial L}{\partial \vec{r}} = -e\vec{\nabla}\left(\varphi - \vec{v}\cdot\vec{A}\right)$$

Both of these equations are satisfied if L is of the form

$$L=\frac{1}{2}m\vec{r}\cdot\vec{r}-e\varphi+e\vec{v}\cdot\vec{A}$$

Or, $$L=\frac{1}{2}mv^2-e\varphi+e\vec{v}\cdot\vec{A}$$

This expression, therefore, gives the required Lagrangian of a particle in an electromagnetic field.

Since, $L=T-V=\frac{1}{2}mv^2-V$, therefore, V is given by

$$V=e\varphi-e\vec{v}\cdot\vec{A}=e\left(\varphi-\vec{v}\cdot\vec{A}\right)$$

7. Figure below shows a solid cylinder with centre G and radius a rolling on the rough inside surface of a fixed cylinder with centre O and radius $b>a$. Find the Lagrange equation of motion and deduce the period of small oscillations about the equilibrium position.

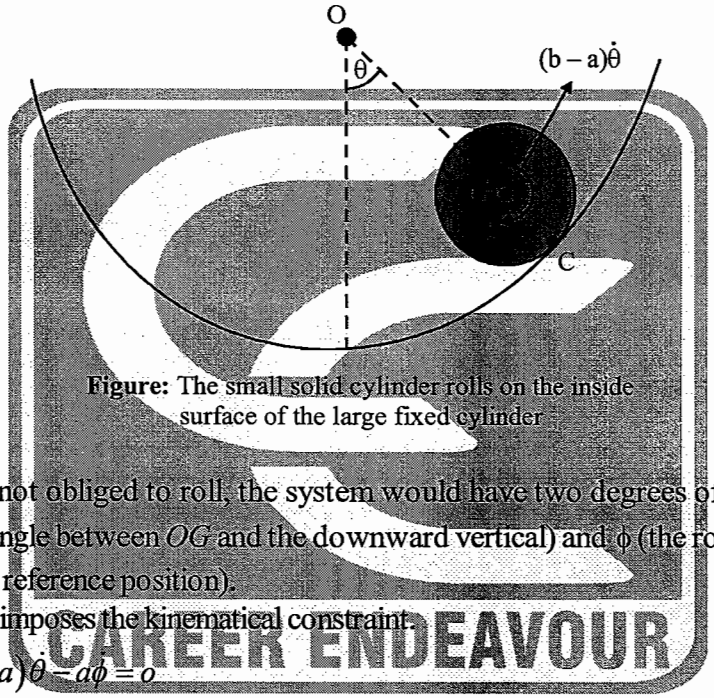

Figure: The small solid cylinder rolls on the inside surface of the large fixed cylinder

Soln. If the cylinder were not obliged to roll, the system would have two degrees of freedom with generalised coordinates θ (the angle between OG and the downward vertical) and ϕ (the rotation angle of the cylinder measured from some reference position).

The rolling condition imposes the kinematical constraint.

$$(b-a)\dot{\theta}-a\dot{\phi}=o$$

This constraint is integrable and is equivalent to the geometrical constraint.

$$(b-a)\theta-a\phi=o$$

on taking $\phi=0$ when $\theta=0$. Thus the rolling cylinder is a standard conservative system with one degree of freedom.

Take θ as the generalised coordinate. Then the **kinetic energy** is given by

$$T=\frac{1}{2}m\left((b-a)\dot{\theta}\right)^2+\frac{1}{2}\left(\frac{1}{2}ma^2\right)\dot{\phi}^2$$

$$=\frac{1}{2}m\left((b-a)\dot{\theta}\right)^2+\frac{1}{2}\left(\frac{1}{2}ma^2\right)\left(\frac{b-a}{a}\right)^2\dot{\theta}=\frac{3}{4}m(b-a)^2\dot{\theta}^2$$

and the potential energy by $V=-mg(b-a)\cos\theta$

There is only one Lagrange equation, namely

$$\frac{d}{dt}\left[\frac{3}{2}m(b-a)^2\dot{\theta}\right]-0=-mg(b-a)\sin\theta$$

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which simplifies to give

$$\ddot{\theta}+\frac{2g}{3(b-a)}\sin\theta=0$$

Interestingly, this equation is identical to the exact equation for the oscillations of a simple pendulum of length $3(b-a)/2$ as obtaind.

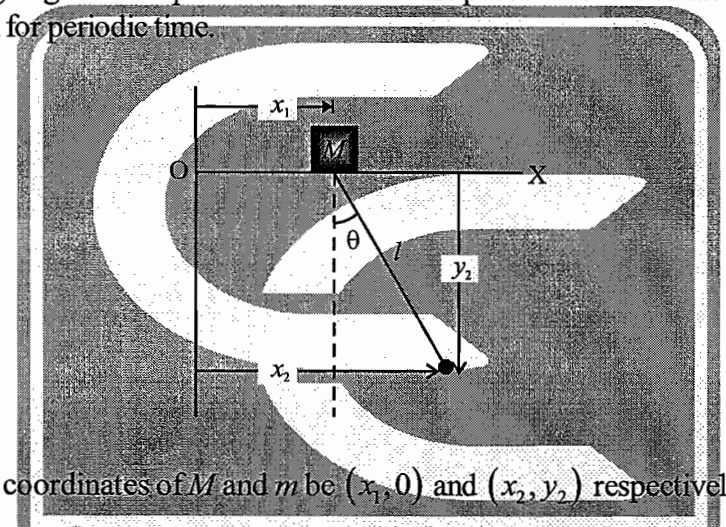
The linearised equation governing small oscillations of the cylinder about $\theta=0$ is is

$$\ddot{\theta}+\frac{2g}{3(b-a)}\theta=0$$

so that the period τ of small oscillation is given by

$$\tau=2\pi\left(\frac{3(b-a)}{2g}\right)^{1/2}$$

8. A pendulum of mass m is attached to a block of mass M. The block slides on a horizontal frictionless surface. Find the Lagrangian and equation of motion of the pendulum. For small amplitude oscillations, derive an expression for periodic time.



Soln. Let at any time t the coordinates of M and m be $(x_1,0)$ and (x_2,y_2) respectively.

Here, $x_2=x_1+\ell\sin\theta$ and $y_2=-\ell\cos\theta$. The suitable generalized coordinates are x_1 and θ.

$$T=\frac{1}{2}M\dot{x}_1^2+\frac{1}{2}m\left(\dot{x}_2^2+\dot{y}_2^2\right)$$

$$=\frac{1}{2}M\dot{x}_1^2+\frac{1}{2}m\left[\left(\dot{x}_1+\ell\dot{\theta}\cos\theta\right)^2+\left(\ell\dot{\theta}\sin\theta\right)^2\right]$$

$$=\frac{1}{2}M\dot{x}_1^2+\frac{1}{2}m\left(\dot{x}_1^2+\ell^2\dot{\theta}^2+2\ell\dot{x}_1\dot{\theta}\cos\theta\right)$$

(because $\dot{x}_2=\dot{x}_1+\ell\cos\theta\,\dot{\theta}$ and $\dot{y}_2=\ell\sin\theta\,\dot{\theta}$)

$$V=-mg\cos\theta$$

Hence, $$L=T-V=\frac{1}{2}(M+m)\dot{x}_1^2+\frac{1}{2}m\ell^2\dot{\theta}^2+m\ell\dot{x}_1\dot{\theta}\cos\theta+mg\ell\cos\theta$$

We see that x_1 is cyclic coordinate, so

Here, $\dfrac{\partial L}{\partial x_1}=0$ and $\dfrac{\partial L}{\partial\dot{x}_1}=(M+m)\dot{x}_1+m\ell\dot{\theta}\cos\theta$ is conserved.

$$\frac{\partial L}{\partial\theta}=m\ell\left(\dot{x}_1\dot{\theta}+g\right)(-\sin\theta)\text{ and }\frac{\partial L}{\partial\dot{\theta}}=m\ell^2\dot{\theta}+m\ell\dot{x}_1\cos\theta$$

Equation of motion in θ is

$$m\ell^2\ddot{\theta} + m\ell\ddot{x}_1\cos\theta + m\ell(-\sin\theta)\dot{\theta}\,\dot{x}_1 - m\ell(-\sin\theta)\dot{x}_1\dot{\theta} + mg\ell\sin\theta = 0$$

Or, $$m\ell^2\ddot{\theta} + m\ell\cos\theta\,\ddot{x}_1 + mg\ell\sin\theta = 0$$

If θ is small, $\sin\theta \approx \theta$ and also $\cos\theta \approx 1$, then

$$m\ell^2\ddot{\theta} + m\ell\ddot{x}_1 + mg\ell\theta = 0$$

Or, $$\ddot{\theta} + \frac{\ddot{x}_1}{\ell} + \frac{g}{\ell}\theta = 0 \qquad \text{... (i)}$$

Equation of motion in x_1 is

$$(M+m)\ddot{x}_1 + m\ell(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = 0$$

For small θ, $(\cos\theta \cong 1, \sin\theta \cong \theta$ and $\dot{\theta}^2\theta$ is negligible$)$

$$(M+m)\ddot{x}_1 + m\ell\ddot{\theta} = 0 \qquad \text{... (ii)}$$

From equations (i) and (ii), we have

$$\ddot{\theta} - \frac{m\ddot{\theta}}{M+m} + \frac{g}{\ell}\theta = 0$$

Hence, $$\ddot{\theta} = -\left[\frac{M+m}{M}\right]\frac{g}{\ell}\theta$$

This is the equation of simple harmonic motion whose period is given by

$$T = 2\pi\sqrt{\frac{\ell}{g}}\sqrt{\frac{M}{M+m}}$$

9. A uniform disc of radius 'a' and mass m, rotates about a fixed axis. A mass less rope is fixed to a points on the out side 'circumference' and leads to mass less spring which is intwin fastened to a fixed point. At a radius $a/2$ another cord is fastenped to a spring which connects to a ma, m. Set up the Lagrange's eq. of the DISC and the mass.

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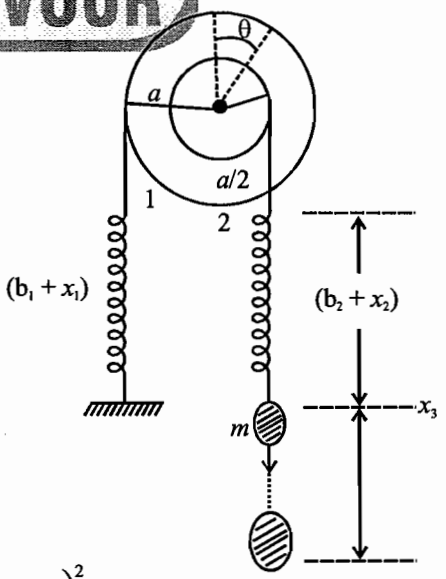
Soln. Let b_1 & b_2 be the original length of the spring 1 & 2. Which the whole system is in eq. and the DISC is stationary.



Let θ be the angle the DISC has twined away from the eq. position. Then extension of the spring – 1 is

$$\boxed{x_1 = a\theta} \qquad \text{.....(1)}$$

If the spring–2 is stretched by a distance x_2 the mass m is low ered by a distance

$$\boxed{x_3 = \frac{a}{2}\theta + x_2} \qquad \text{.....(2)}$$

K.E. $\quad T =$ K.E. of DISC + K.E. of mass

$$= \frac{1}{2}(I\dot{\theta})^2 + \frac{1}{2}m\dot{x}_3^2 = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}m\left(\frac{a}{2}\dot{\theta} + \dot{x}_2\right)^2$$

P.E. $\quad V = \frac{1}{2}k x_1^2 + \frac{1}{2}k x_2^2 - mgx_3$.

Hence the Lagrangian for the system is

$$L = T - V$$

$$L = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}m\left(\frac{a^2}{4}\dot{\theta}^2 + a\dot{\theta}\dot{x}_2 + \dot{x}_2^2\right) - \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 + mg\left(\frac{a}{2}\theta + x_2\right)$$

Lagrangian eq. of motion, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \left(\frac{\partial L}{\partial x_2}\right) = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right)$

Or $\quad \frac{d}{dt}\left[\frac{1}{2}ma\theta + m\dot{x}_2\right] - \left[-kx_2 + mg\right] = 0 \quad \Rightarrow \quad m\ddot{x}_2 + kx_2 + \frac{1}{2}m(a\ddot{\theta} - 2g) = 0$

Or $\quad \boxed{\ddot{x} + \left(\frac{k}{m}\right)x_2 + \frac{1}{2}(a\ddot{\theta} - 2g) = 0} \quad \Rightarrow \quad \frac{d}{dt}\left[\frac{1}{2}ma^2\dot{\theta} + \frac{1}{2}ma^2 + \dot{\theta}\max_2\right] - \left[-ka^2\theta + \frac{mga}{2}\right]$

Or $\quad \frac{3}{4}ma^2\ddot{\theta} + Ra^2\theta + \frac{1}{2}ma(\ddot{x}_2 - g) = 0$

Or $\quad \boxed{\ddot{\theta} + \left(\frac{4}{3}\frac{k}{m}\right)\theta + \frac{2}{3a}(\ddot{x}_2 - g) = 0}$

Which are the required equation of motion.

10. Set up the lagrangian for the following system, where the pully is massless.

Soln. $\quad T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$

$$V = -m_1gx_1 - m_2gx_2 + \frac{1}{2}kx_1^2$$

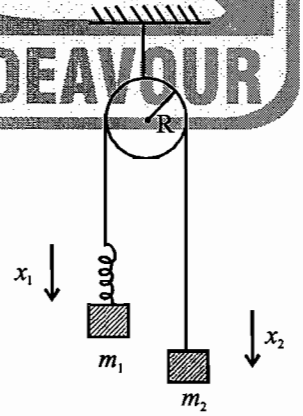
$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + m_1gx_1 + m_2gx_2 - \frac{1}{2}kx_1^2$$

Thus, equation of motion is $= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \left(\frac{\partial L}{\partial x_1}\right) = 0$

And $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \left(\frac{\partial L}{\partial x_2}\right) = 0$

$\Rightarrow \quad m_1\ddot{x}_1 - m_1g + kx_1 = 0$

Say, $m_2\ddot{x}_2 - m_2g = 0$



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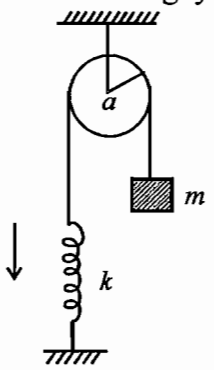
11. Set up the lagrangian for the following system, the disk also have mass 'm'.



Soln. Let θ be the angle one disc has turned away from one equilibrium position. Then extension of the spring $x_1 = a\theta$

Mass m is lowered by a distance $x_2 = a\theta$

$$\therefore \quad T = \frac{1}{2}m(a\dot{\theta})^2 + \frac{1}{2}I\dot{\theta}^2$$

$$V = -mga\theta + \frac{1}{2}k\theta^2a^2$$

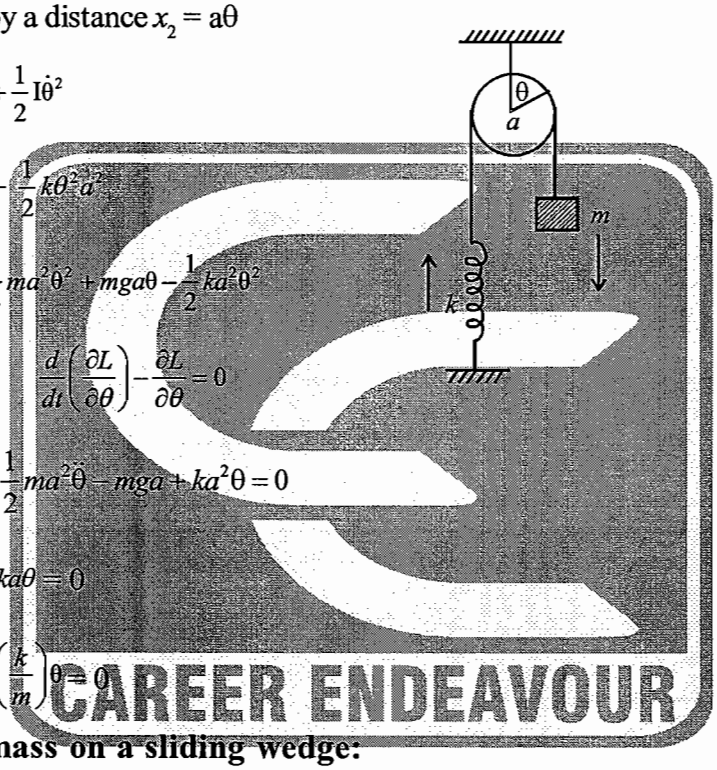
$$L = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{4}ma^2\dot{\theta}^2 + mga\theta - \frac{1}{2}ka^2\theta^2$$

Eq. of motion is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$$\Rightarrow \quad ma^2\ddot{\theta} + \frac{1}{2}ma^2\ddot{\theta} - mga + ka^2\theta = 0$$

$$\Rightarrow \quad \frac{3}{2}ma\ddot{\theta} - mg + ka\theta = 0$$

$$\Rightarrow \quad \ddot{\theta} - \frac{2}{3}g/a + \frac{2}{3}\left(\frac{k}{m}\right)\theta = 0$$



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A sliding point mass on a sliding wedge:

Consider the situation depicted in figure, in which a point object of mass m slides frictionlessly along a wedge of opening angle α. The wedge itself slides frictionlessly along a horizontal surface, and its mass is M. We choose as generalized coordinates the horizontal position X of the left corner of the wedge, and the horizontal distance x from the left corner to the sliding point mass. The vertical coordinate of the sliding mass is then $y = x\tan\alpha$.

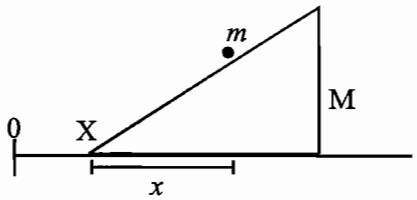

Figure : A wedge of mass M and opening angle a slides frictionlessly along a horizontal surface, while a small object of mass m slides frictionlessly along the wedge.

where the horizontal surface lies at $y = 0$. With the generalized coordinates, the kinetic energy is

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left(\dot{X} + \dot{x}\right)^2 + \frac{1}{2}m\dot{y}^2$$

$$= \frac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{x} + \frac{1}{2}m\left(1+\tan^2\alpha\right)\dot{x}^2$$

The potential energy is simply,

$$U = mgy = mgx\tan\alpha$$

Thus, the Lagrangian is

$$L = \frac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{x} + \frac{1}{2}m\left(1+\tan^2\alpha\right)\dot{x}^2 - mgx\tan\alpha$$

and the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) = \frac{\partial L}{\partial X} \Rightarrow (M+m)\ddot{X} + m\ddot{x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \Rightarrow m\ddot{X} + m\left(1+\tan^2\alpha\right)\ddot{x} = -mg\tan\alpha$$

At this point we can use the first of these equations to write

$$\ddot{X} = -\frac{m}{M+m}\ddot{x}\partial$$

Substituting this into the second equation, we obtain the constant accelerations.

$$\ddot{x} = -\frac{(M+m)g\sin\alpha\cos\alpha}{M+m\sin^2\alpha}, \quad \ddot{X} = \frac{mg\sin\alpha os\alpha}{M+m\sin^2\alpha}$$

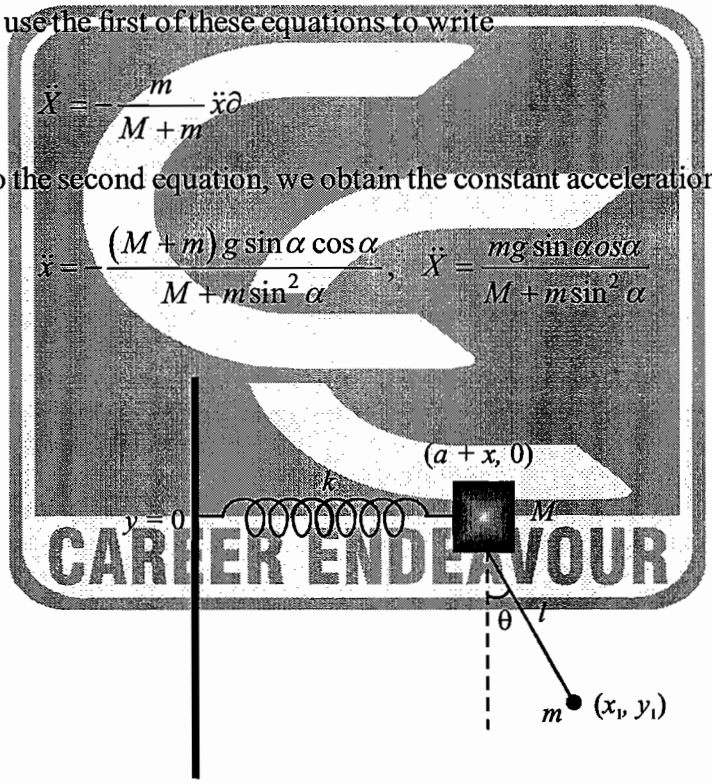

***Figure** : The spring-pendulum system*

A pendulum attached to a mass on a spring:

Conside next the system depicted in figure in above in which a mass M moves horizontally while attached to a spring of spring constant *k*. Hanging from this mass is a pendulum of arm length *l* and bob mass *m*.

A convenient set of generalized coordinates is (x,θ), where x is the displacement of the mass M relative to the equilibrium extension a of the spring, and θ is the angle the pendulum arm makes with respect to the vertical. Let the Cartesian coordinates of the pendulum bob be (x_1, y_1). Then,

$$x_1 = a + x + \ell\sin\theta, \qquad y_1 = -\ell\cos\theta$$

The kinetic energy is,

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[\left(\dot{x} + \ell\cos\theta\dot{\theta}\right)^2 + \left(\ell\sin\theta\,\dot{\theta}\right)^2\right]$$

$$= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2 + m\ell\cos\theta\dot{x}\dot{\theta}$$

and the potential energy is,

$$U = \frac{1}{2}kx^2 + mgy_1 = \frac{1}{2}kx^2 - mg\ell\cos\theta$$

Thus, $$L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2 + m\ell\cos\theta\dot{x}\dot{\theta} - \frac{1}{2}kx^2 + mg\ell\cos\theta$$

The canonical moments are

$$p_x = \frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + m\ell\cos\theta\,\dot{\theta}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\ell\cos\theta\dot{x} + m\ell^2\dot{\theta}$$

and the canonical forces are

$$F_x = \frac{\partial L}{\partial x} = -kx$$

$$F_\theta = \frac{\partial L}{\partial \theta} = -m\ell\sin\theta\dot{x}\,\dot{\theta} - mg\ell\sin\theta$$

The equations of motion then yield

$$(M+m)\ddot{x} + m\ell\cos\theta\,\ddot{\theta} - m\ell\sin\theta\,\dot{\theta}^2 = -kx$$

$$m\ell\cos\theta\ddot{x} + m\ell^2\ddot{\theta} = -mg\ell\sin\theta$$

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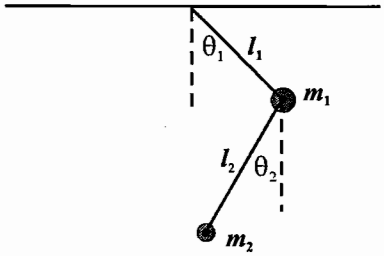

Figure :The double pendulum, with generalized coordinates θ_1 and θ_2. All motion is confined to a single plane.

The double pendulum:

As yet another example of the generalized coordinate approach to Lagrangian dynamics, consider the pendulum system, sketched in figure above. We choose as generalized coordinates the two angles θ_1 and θ_2. In order to evaluate the Lagrangian, we must obtain the kinetic and potential energies in terms of generalized coordinates $\{\theta_1, \theta_2\}$ and their corresponding velocities $\{\theta_1, \theta_2\}$.

In cartesian coordinates,

$$T = \frac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

$$U = m_1 g y_1 + m_2 g y_2$$

We therefore express the Cartesian coordinates $\{x_1, y_1, x_2, y_2\}$ in terms of generalized coordinates $\{\theta_1, \theta_2\}$.

$$x_1 = \ell_1 \sin\theta_1 \qquad x_2 = \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2$$

$$y_1 = -\ell_1 \cos\theta_1 \qquad y_2 = -\ell_1 \cos\theta_1 - \ell_2 \cos\theta_2$$

Thus, the velocities are

$$\dot{x}_1 = \ell_1\dot{\theta}_1 \cos\theta_1 \qquad \dot{x}_2 = \ell_1\dot{\theta}_1 \cos\theta_1 + \ell_2\dot{\theta}_2 \cos\theta_2$$

$$\dot{y}_1 = \ell_1\dot{\theta}_1 \sin\theta_1 \qquad \dot{y}_2 = \ell_1\dot{\theta}_1 \sin\theta_1 + \ell_2\dot{\theta}_2 \sin\theta_2$$

Thus,

$$T = \frac{1}{2}m_1\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left\{\ell_1^2\dot{\theta}_1^2 + 2\ell_1\ell_2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + \ell_2^2\dot{\theta}_2^2\right\}$$

$$U = -m_1 g\ell_1\cos\theta_1 - m_2 g\ell_1\cos\theta_1 - m_2 g\ell_2\cos\theta_2$$

And

$$L = T - U = \frac{1}{2}(m_1 + m_2)\ell_1^2\dot{\theta}_1^2 + m_2\ell_1\ell_2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}m_2\ell_1^2\dot{\theta}_2^2 + (m_1 + m_2)g\ell_1\cos\theta_1 + m_2 g\ell_2\cos\theta_2$$

The generalized (canonical) momenta are

$$p_1 = \frac{\partial L}{\partial\dot{\theta}_1} = (m_1 + m_2)\ell_1^2\dot{\theta}_1 + m_2\ell_1\ell_2\cos(\theta_1 - \theta_2)\dot{\theta}_2$$

$$p_2 = \frac{\partial L}{\partial\dot{\theta}_2} = m_2\ell_1\ell_2\cos(\theta_1 - \theta_2)\dot{\theta}_1 + m_2\ell_2^2\dot{\theta}_2$$

CAREER ENDEAVOUR

Spherical pendulum:

A spherical pendulum is essentially a simple pendulum which is free to move through the entire space about the point of suspension. The bob of such a pendulum moves on the surface of a sphere whose radius is equal to the length of the pendulum. It is obvious that spherical polar coordinates are most suitable to locate the position of the bob of the pendulum. Moreover, since l is a constant, only θ and φ need be specified.

The velocity of the bob is given by

$$v = \ell\dot{\theta}\,\hat{e}_\theta + \ell\sin\theta\,\dot{\varphi}\hat{e}_\varphi$$

Hence, the kinetic energy of the pendulum is given by

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\ell^2\dot{\theta}^2 + \ell^2\sin^2\theta\dot{\varphi}^2\right)$$

Potential energy V due to gravity is $V = -mg\ell\cos\theta$;

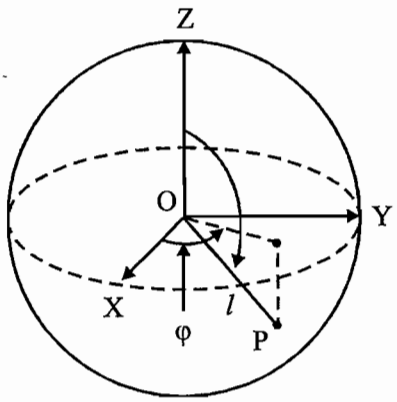

Figure : Spherical pendulum

It is masured with reference to the horizontal plane $z = 0$.

Hence, the Lagrangian function L is given by

$$L = T - V = \frac{1}{2} m\ell^2 \dot{\theta}^2 + \frac{1}{2} m\ell^2 \sin^2 \theta \dot{\varphi}^2 + mg\ell \cos\theta$$

Thus, the Lagrangian equations corresponding to coordinates θ and φ are given by

$$\frac{d}{dt}\left(m\ell^2\dot{\theta}\right) - m\ell^2 \sin\theta\cos\theta\,\dot{\varphi}^2 - mg\ell\sin\theta = 0 \text{ and } \frac{d}{dt}\left(m\ell^2 \sin^2\theta\,\dot{\varphi}\right) = 0$$

Variable φ is ignorable and hence the corresponding momentum is conserved. Thus,

$$p_\varphi = m\ell^2 \sin^2\theta\,\dot{\varphi} = \text{constant}$$

Substituting the value of φ in terms of p_φ, we get the one-dimensional equation of motion in θ, viz.

$$m\ell^2\ddot{\theta} = mg\ell\sin\theta + \frac{p_\varphi^2 \cos\theta}{m\ell^2 \sin^3\theta}$$

We can find the total energy of the pendulum as follows.

Since $\frac{\partial L}{\partial t} = 0$, total energy E is given by equation.

Noether's Theorem:

Conservation of linear momentum and angular momentum are particular instances of an abstract result known as ***Noether's theorem.*** In each of the above cases, there is a one-parameter family of mapping $\left(\mathfrak{M}^\lambda\right)$, parameterised by a real variable λ, that act on the configuration space, that is,

$$q \xrightarrow{\mathfrak{M}^\lambda} q^\lambda$$

In each case $\lambda = 0$ corresponds to the identify mapping ($q \to q$), and in each case the potential energy $V(q)$ is invariant under $\{\mathfrak{M}^\lambda\}$, that is

$$V\left(q^\lambda\right) = V(q)$$

CAREER ENDEAVOUR

From these facts, we were able to prove that, in each case, a certain momentum component was a constant of the motion.

This idea was generalised by Noether to apply to any Lagrangian system and any family of mapping $\{\mathfrak{M}^\lambda\}$, provided that the Lagrangian L is invariant under $\{\mathfrak{M}^\lambda\}$ in the sense that

$$L\left(q^\lambda, \dot{q}^\lambda, t\right) = L(q, \dot{q}, t)$$

for all λ. In this formula, q^λ is a known function of the variables q and λ (as defined by the mapping $\mathfrak{M}^\lambda$), but we have not yet said what we mean by $\dot{q}^\lambda$. This is however defined in the following commonsense way: let λ be fixed and let q^λ be the image point of a typical point q. Suppose now that the point q has velocity $\dot{q}$ in the configuration space. This motion of q imparts a velocity to the image point q^λ, and it is this velocity that we call $\dot{q}^\lambda$. This definition is expressed by the formula.

$$\dot{q}^\lambda = \sum_{j=1}^{n} \frac{\partial q^\lambda}{\partial q_j} \dot{q}_j$$

from which we see that $\dot{q}^\lambda$ is a known function of the variables $q, \dot{q}$ and λ.

Previous Year CSIR-NET Questions

1. The Lagrangian of a particle of charge e and mass m in applied electric and magnetic fields is given by $L = \frac{1}{2}m\vec{v}^2 + e\vec{A}.\vec{v} - e\phi$, where $\vec{A}$ and ϕ are the vector and scalar potentials corresponding to the magnetic and electric fields, respectively. Which of the following statements is correct?

(a) The canonically conjugate momentum of the particle is given by $\vec{p} = m\vec{v}$

(b) The Hamiltonian of the particle is given by $H = \frac{\vec{p}^2}{2m} + \frac{e}{m}\vec{A}.\vec{p} + e\phi$

(c) L remains unchanged under a gauge transformation of the potentials.

(d) Under a gauge transformation of the potentials, L changes by the total time derivative of a function of $\vec{r}$ and t.

Soln. $L = \frac{1}{2}m\vec{v} - e\phi + e\vec{A}.\vec{v}$

We know that $\vec{E}$ and $\vec{B}$ fields are in variant under gauge transformation.

$$\vec{A}(\vec{x},t) \to \vec{A}' = \vec{A} + \vec{\nabla}\lambda(\vec{x},t); \qquad \varphi(\vec{x},t) \to \varphi' = \varphi - \frac{\partial\lambda}{\partial t}(\vec{x},t)$$

where $\lambda(\vec{x},t)$ is an arbitrary scalar function.

$$\therefore \qquad L \to L' = L + e\left(\frac{\partial\lambda}{\partial t}(\vec{x},t) + \vec{v}.\vec{\nabla}\lambda(\vec{x},t)\right)$$

The expression in bracket is just the total time derivative of $\lambda(\vec{x},t)$.

If we add a total time derivative of a function of $\vec{x}$ and t to the Lagrangian, the equations of motion do not change.

Correct answer : (d)

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2. The Hamiltonian o f a system with n degrees of freedom is given by $H(q_i, \ldots\ldots, q_n; p_i, \ldots\ldots, p_n; t)$, with an explicit dependence on the time t. Which of the following is correct?

(a) Different phase trajectories cannot intersect each other

(b) H always represents the total energy of the system and is a constant of the motion.

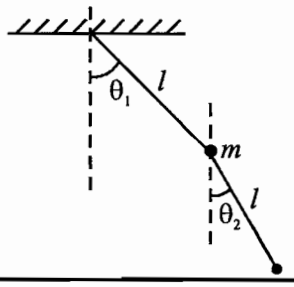
(c) The equations $\dot{q}_i = \partial H / \partial p_i, \dot{p}_i = -\partial H / \partial q_i$ are not valid since H has explicit time dependence.

(d) Any initial volume element in phase space remains unchanged in magnitude under time evolution.

Soln. According to Liouville's theorem, the phase volume occupied by a collection of system evolve according to Hamilton's equation of motion, and will be preserved in time.

Correct answer: (d)

3. A double pendulum consists of two point masses m attached by massless strings of length *l* as shown in the figure:



The kinetic energy of the pendulum is :

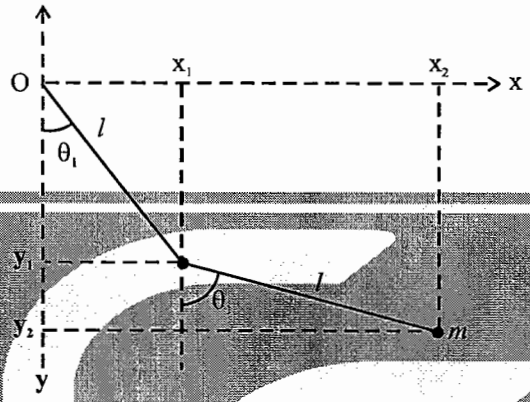
(a) $\frac{1}{2}m\ell^2\left[\dot{\theta}_1^2+\dot{\theta}_2^2\right]$ (b) $\frac{1}{2}m\ell^2\left[2\dot{\theta}_1^2+\dot{\theta}_2^2+2\dot{\theta}_1\dot{\theta}_2\cos\left(\dot{\theta}_1-\dot{\theta}_2\right)\right]$

(c) $\frac{1}{2}m\ell^2\left[\dot{\theta}_1^2+2\dot{\theta}_2^2+2\dot{\theta}_1\dot{\theta}_2\cos\left(\dot{\theta}_1-\dot{\theta}_2\right)\right]$ (d) $\frac{1}{2}m\ell^2\left[2\dot{\theta}_1^2+\dot{\theta}_2^2+2\dot{\theta}_1\dot{\theta}_2\cos\left(\dot{\theta}_1+\dot{\theta}_2\right)\right]$

Soln. $x_1=\ell\sin\theta_1,\quad x_2=\ell\sin\theta_1+\ell\sin\theta_2$

$y_1=-\ell\cos\theta_1,\qquad y_2=-\ell\cos\theta_1-\ell\cos\theta_2$

$\dot{x}_1=\ell\cos\theta_1\,\dot{\theta}_1,\qquad \dot{y}_1=\ell\sin\theta_1\dot{\theta}_1$



$\dot{x}_2=\ell\cos\theta_1\,\dot{\theta}_1+\ell\cos\theta_2\dot{\theta}_2;\quad \dot{y}_2=\ell\sin\theta_1\,\dot{\theta}_1+\ell\sin\theta_2\dot{\theta}_2,$

$$T=\frac{m}{2}\left(\dot{x}_1^2+\dot{y}_1^2\right)+\frac{m}{2}\left(\dot{x}_2^2+\dot{y}_2^2\right)$$

$$=\frac{m}{2}\dot{\theta}_1^2\ell^2+\frac{m}{2}\left[\dot{\theta}_1^2\ell^2+\dot{\theta}_2^2\ell^2+2\ell^2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1-\theta_2\right)\right]$$

$$=\frac{m\ell^2}{2}\left[2\dot{\theta}_1^2+\dot{\theta}_2^2+2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1-\theta_2\right)\right]$$

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Correct answer is (b).

4. A particle of mass 'm' moves inside a bowl. If the surface of the bowl is given by the equation $z=\frac{1}{2}a\left(x^2+y^2\right)$, where a is a constant, the Lagrangian of the particle is:

(a) $\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2-gar^2\right)$ (b) $\frac{1}{2}m\left[\left(1+a^2r^2\right)\dot{r}^2+r^2\dot{\phi}^2\right]$

(c) $\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\dot{\phi}^2-gar^2\right)$ (d) $\frac{1}{2}m\left[\left(1+a^2r^2\right)\dot{r}^2+r^2\dot{\phi}^2-gar^2\right]$

Soln. $z=\frac{1}{2}a\left(x^2+y^2\right)=\frac{1}{2}a\,r^2\qquad\Rightarrow\dot{z}=ar\dot{r}$

$$K.E.=T=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)$$

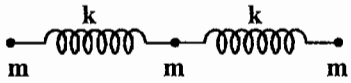
$$=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\varphi}^2+\dot{z}^2\right)$$

$$=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\varphi}^2+a^2r^2\dot{r}^2\right) \qquad \text{P.E.} = V = mgz = \frac{1}{2}mgar^2$$

$$\therefore \quad L = T - V = \frac{1}{2}m\left(\dot{r}^2+r^2\dot{\varphi}^2+a^2r^2\dot{r}^2-gar^2\right) = \frac{1}{2}m\left[\left(1+a^2r^2\right)\dot{r}^2+r^2\dot{\varphi}^2-gar^2\right]$$

Correct answer is (d).

5. Three particles of equal mass 'm' are connected by two identical massless springs of stiffness constant 'k' as shown in the figure.



If x_1, x_2 and x_3 denote the displacements of the masses from their respective equilibrium positions, the potential energy of the system is:

(a) $\frac{1}{2}k\left(x_1^2+x_2^2+x_3^2\right)$ (b) $\frac{1}{2}k\left[x_1^2+x_2^2+x_3^2-x_2\left(x_1+x_3\right)\right]$

(c) $\frac{1}{2}k\left[x_1^2+2x_2^2+x_3^2+2x_2\left(x_1+x_3\right)\right]$ (d) $\frac{1}{2}k\left[x_1^2+2x_2^2+x_3^2-2x_2\left(x_1+x_3\right)\right]$

Soln. Potential energy $=\frac{1}{2}k\left(x_2-x_1\right)^2$ first spring

Potential energy $=\frac{1}{2}k\left(x_3-x_2\right)^2$ second spring

Potential energy of the system,

$$=\frac{1}{2}k\left(x_2-x_1\right)^2+\frac{1}{2}k\left(x_3-x_2\right)^2=\frac{1}{2}k\left[x_1^2+x_2^2-2x_1x_2+x_2^2+x_3^2-2x_2x_3\right]$$

$$=\frac{1}{2}k\left[x_1^2+2x_2^2+x_3^2-2x_2\left(x_1+x_3\right)\right]$$

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Correct option is (d)

6. The Lagrangian of a particle of mass m moving in one dimensions is given by

$$L=\frac{1}{2}m\dot{x}^2-bx$$

where b is a positive constant. The coordinate of the particle x(t) at time t is given by: (in the following c_1 and c_2 are constants)

(a) $-\frac{b}{2m}t^2+c_1t+c_2$ (b) $c_1t + c_2$

(c) $c_1\cos\left(\frac{bt}{m}\right)+c_2\sin\left(\frac{bt}{m}\right)$ (d) $c_1\cosh\left(\frac{bt}{m}\right)+c_2\sinh\left(\frac{bt}{m}\right)$

Soln. $L=\frac{1}{2}m\dot{x}^2-bx = T - V$

$\therefore V = bx$

This is same as uniform gravitational potential. Therefore, solution must be of the form.

$S = S_0 + ut + \frac{1}{2}at^2$

Correct option is (a)

7. A particle moves in a potential $V = x^2 + y^2 + \frac{z^2}{2}$. Which component (s) of the angular momentum is / are constant (s) of motion?

(a) none (b) L_x, L_y and L_z (c) only L_x and L_y (d) only L_z

Soln. Given $V(x,y,z) = x^2 + y^2 + \frac{z^2}{2}$

In polar co-ordinate (spherical polar)

$$V = r^2\sin^2\theta + \frac{r^2\cos^2\theta}{2} \qquad \begin{bmatrix} x = r\cos\phi\sin\theta \\ y = r\sin\phi\sin\theta \\ z = r\cos\theta \end{bmatrix}$$

Therefore, Lagrangian of system is

$L = T - V$

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - r^2\sin^2\theta - \frac{r^2\cos^2\theta}{2}$$

ϕ is cyclic, therefore p_ϕ is constant of motion.

p_ϕ is equal to L_z. Therefore, L_z is constant of motion.

Correct answer is (d)

CAREER ENDEAVOUR

Deduction of Hamilton's equations of motion for Holonomic Conservative system.

(a) From Lagrange's equations : We have as Lagrange's equations for a holonomic conservative system of n particles having f degrees of freedom.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$$

Introducing geeralised momenta, $p_k = \frac{\partial L}{\partial \dot{q}_k}$ we have Lagrange's equations

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \quad or \quad \dot{p}_k = \frac{\partial L}{\partial q_k}$$

Let us now introduce a function H and call it the Hamiltonian function (or simply Harmiltonian) of the system after the name of pioneer worker in this direction and define it as a function of the generalised coordinations and momenta of the system, as well as of time in the most general way and is of the form in the most general way,

$$H(q,p,t) = \sum_{k=1}^{k=f} p_k\dot{q}_k - L(q,\dot{q},t) \qquad \ldots (i)$$

where all the $\dot{q}$ and the Lagrangian function L are expressed in terms of generalised coordinates and momenta.

In a conservative system the Lagrangian does not involve the time explicitly and so the Hamiltonian is a function of generalised momenta and coordinates and not of time. Therefore, for a conservative holonomic system, $H = H(q, p)$

By the rule of partial differentiation,

$$dH = \sum_{k=1}^{k=f}\left(\frac{\partial H}{\partial q_k}dq_k + \frac{\partial H}{\partial p_k}dp_k\right) \quad \text{... (ii)}$$

Differentiating the expression for H i.e. equation (i)

$$dH = d\left(\sum_{k=1}^{k=f} p_k\dot{q}_k - L\right) = d\sum_{k=1}^{k=f} p_k\dot{q}_k - dL$$

Or,
$$dH = \sum_{k=1}^{k=f}\left(p_k d\dot{q}_k + \dot{q}_k dp_k\right) - \sum_{k=1}^{k=f}\left(\frac{\partial L}{\partial q_k}dq_k + \frac{\partial L}{\partial \dot{q}_k}d\dot{q}_k\right)$$

By Lagrange's equation, $\frac{\partial L}{\partial q_k} = \dot{p}_k$

$$\therefore \quad dH = \sum_{k=1}^{k=f}\left(p_k d\dot{q}_k + \dot{q}_k dp_k - \dot{p}_k\, dq_k - p_k d\dot{q}_k\right) \quad \left(\because \frac{\partial L}{\partial \dot{q}_k} = p_k\right)$$

$$= \sum_{k=1}^{k=f}\left(\dot{q}_k dp_k - \dot{p}_k dq_k\right) \quad \text{... (iii)}$$

From equation (ii) and (iii), we have

$$\sum_{k=1}^{k=f}\left(\frac{\partial H}{\partial q_k}dq_k + \frac{\partial H}{\partial p_k}dp_k\right) = \sum_{k=1}^{k=f}\left(\dot{q}_k dp_k - \dot{p}_k dq_k\right)$$

Comparing coefficients of dp_k and dq_k

$$\left.\begin{aligned}\dot{q}_k &= \frac{\partial H}{\partial p_k}\\ \dot{p}_k &= -\frac{\partial H}{\partial q_k}\end{aligned}\right\}$$

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These equations are called Hamilton's equations or Hamilton's canonical equations. There are $2f$ Hamilton's equations that describe a system of 'f' degrees of freedom. Each Hamilton equation is of the first order and first degree.

Physical significance of the Hamiltonian:

Firstly, it is a constant of motion of all systems having its Lagrangian and kinetic energy independent of time [such as a free system, system of stationary (scleronomous) constraints, conservative system] and secondly, it represents the total energy of such systems. Let us prove it for such systems.

Proof : Since in the most general way Hamiltonian H is a function of q's, p's and t we have by the rule of partial differentiation,

$$\frac{dH}{dt} = \sum_{k=1}^{k=f}\left(\frac{\partial H}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial H}{\partial p_k}\frac{dp_k}{dt}\right) + \frac{\partial H}{\partial t}\frac{dt}{dt}$$

where f = the number of degrees of freedom.

$$= \sum_{k=1}^{k=f} \left(\frac{\partial H}{\partial q_k} \right) \left(\frac{\partial H}{\partial q_k} \dot{q}_k + \frac{\partial H}{\partial p_k} \dot{p}_k \right) + \frac{\partial H}{\partial t}$$

By Hamilton's equations we have $\frac{\partial H}{\partial q_k} = -\dot{p}_k$ and $\frac{\partial H}{\partial p_k} = \dot{q}_k$

$$\therefore \quad \frac{dH}{dt} = \sum_{k=1}^{k=f} \left(-\dot{p}_k \dot{q}_k + \dot{q}_k \dot{p}_k \right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

Since, $H = \sum_{k=1}^{k=f} p_k \dot{q}_k - L$ and L is independent of time, H is also independent of time, that is, $\frac{\partial H}{\partial t} = 0$

$$\therefore \quad \frac{dH}{dt} = 0 \text{ or } H = a \text{ constant.}$$

When the kinetic energy (T) of a system is independent of time, it is a homogeneous quadratic funtion of generalised velocities. So, by Euler theorem on homogeneous functions, we have

$$\sum_{k=1}^{k=f} \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = 2T$$

But, $L = T - V$, $\therefore \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k}$ (Since, V does not depend on $\dot{q}_k$)

$$\therefore \quad \sum_{k=1}^{k=f} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} = 2T$$

But $\frac{\partial L}{\partial \dot{q}_k} = p_k$; $\therefore \sum_{k=1}^{k=f} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} = 2T$

$$\therefore \quad H = \sum_{k=1}^{k=f} p_k \dot{q}_k - L = 2T - (T - V) = T + V = E, \text{ total energy of the system.}$$

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EXAMPLES

1. Obtain the Hamiltonian function of a compound pendulum and hence obtain the equation describing its motion.

Soln. The pendulum is a harmonic conservative system. It is holonomic because the constraint is on the position of the pendulum in the sense that it must vibrate in its own plane and conservative because the gravitational force acting on it is conservative. With reference to a frame fixed at the point of suspension with downward vertical as the reference line, we have

$$T = \frac{1}{2} I \dot{\theta}^2$$

where I is the moment of inertia of the pendulum about the axis passing through the point suspension and perpendicular to its plane.

$$V = -mg\ell \cos\theta$$

where ℓ is the distance of the centre of gravity from the centre of suspension.

$$\therefore \quad H = T + V = \frac{1}{2} I \dot{\theta}^2 - mg\ell \cos\theta$$

Now, $p_\theta = \frac{\partial T}{\partial \dot{\theta}} = I\dot{\theta}$

$\therefore \quad H = \frac{1}{2} I \frac{p_\theta^2}{I^2} - mg\ell\cos\theta$

Or, $H = \frac{p_\theta^2}{I^2} - mg\ell\cos\theta$

$$\frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{I} \text{ and } \frac{\partial H}{\partial p_\theta} = -mg\ell(-\sin\theta) = mg\ell\sin\theta$$

By Hamilton's general equations $\left(\dot{p}_k = -\frac{\partial H}{\partial q_k} \text{ and } \dot{q}_k = \frac{\partial H}{\partial p_k}\right)$

We have here $\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mg\ell\sin\theta$ and $\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{I}$

These are the two equations describing the motion of a compound pendulum. However, these two equtaions can be combined into a signle equation. Differentiating the second equation and eliminating $\dot{p}_\theta$, we have

$$\ddot{\theta} = \frac{\dot{p}_\theta}{I}\sin\theta \text{ ; or } \ddot{\theta} + \frac{mgl}{I}\sin\theta = 0$$

This is the required single equation that describes the motion of a compound pendulum.

2. A particle moves in the xy plane under the influence of a central force depending on its distance from the origin.
(a) Set up the Hamiltonian for the system
(b) Obtain Hamilton's equations of motion

Soln. **(a)** T (kinetic energy of of the particle) $= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$ $\left(\because I = mr^2\right)$

$\therefore \quad H = T + V = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r)$

Now, $p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r}$ and $p_\theta = \frac{\partial T}{\partial \dot{\theta}} = I\dot{\theta} = mr^2\dot{\theta}$

$\therefore \quad H = \frac{1}{2}m\cdot\frac{p_r^2}{m^2} + \frac{1}{2}mr^2\frac{p_\theta^2}{m^2r^4} + V(r)$

Or, $H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r)$

(b) We have from Hamilton's equation,

$$\left(\dot{p}_k = -\frac{\partial H}{\partial q_k} \text{ and } \dot{q}_k = \frac{\partial H}{\partial p_k}\right)$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -\left(-\frac{p_\theta^2}{mr^3} + \frac{\partial V}{\partial r}\right) = \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r} \qquad \ldots (i)$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \qquad \text{... (ii)}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \qquad \text{... (iii)}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \qquad \text{...(iv)}$$

Equations (i), (ii), (iii) and (iv) are the required Hamilton's equations. However they can be combined to reduce the number of equations describing the motion. Equations (i) and (ii) combine into (after elimination of $\dot{p}_r$)

$$\ddot{r} = \frac{p_\theta^2}{m^2 r^3} - \frac{1}{m}\frac{\partial V}{\partial r} = \frac{m^2 r^4 \dot{\theta}^2}{m^2 r^2} - \frac{1}{m}\frac{\partial V}{\partial r}$$

and (iii) and (iv) combine into (after elimination of $\dot{p}_\theta$) $\ddot{\theta} = 0$

Thus the equations describing motion of the particle are

$$\ddot{r} - r\dot{\theta}^2 = -\frac{1}{m}\frac{\partial V}{\partial r} \text{ and } \ddot{\theta} = 0$$

Or, $$\ddot{r} - r\dot{\theta}^2 = -\frac{1}{m}\frac{\partial V}{\partial r} \text{ and } \dot{\theta} = \frac{p_\theta}{mr^2} = \frac{a \text{ constant}}{mr^2}$$

$$\left(\because p_\theta = a \text{ constant from } (iii)\right)$$

Or, $$mr^2\dot{\theta} = a \text{ constant}$$

3. The Hamiltonian of a particle of unit mass moving in the xy-plane is given to be: **[NET June 2011]**

$H = xp_x - yp_y - \frac{1}{2}x^2 + \frac{1}{2}y^2$ in suitable units. The initial values are given to be (x(0), y(0)) = (1, 1) and $\left(p_x(0), p_y(0)\right) = \left(\frac{1}{2}, -\frac{1}{2}\right)$. During the motion, the curves traced out by the particles in the xy-plane and the $p_x p_y$-plane are

(a) Both straight lines
(b) A straight line and a hyperbola respectively
(c) A hyperbola an ellipse, respectively
(d) Both hyperbolas

Soln. $H = xp_x - yp_y - \frac{x^2}{2} - \frac{y^2}{2}$

Curves in $x-y$ and $p_x - p_y$ plane represent relation between x and y and p_x and p_y respectively.

Hamilton's equations:

$$\dot{x} = \frac{\partial H}{\partial p_x} = x \quad \Rightarrow \dot{x} = x \quad \Rightarrow \frac{dx}{dt} = x \quad \Rightarrow \int_1^x \frac{dx}{x} = \int_0^t dt \quad \Rightarrow x = e^t \qquad \ldots(i)$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = -y \quad \Rightarrow \dot{y} = -y \quad \Rightarrow \frac{dy}{dt} = -y \quad \Rightarrow \int_1^y \frac{dy}{y} = -\int_0^t dt \quad \Rightarrow y = e^{-t} \qquad \ldots(ii)$$

Multiplying (i) and (ii) we get, $xy = 1$, this is equation of rectangular hyperbola in x–y plane.

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Hamiltan's equations:

$$\dot{p}_x = -\frac{\partial H}{\partial_x} = -p_x + x \Rightarrow \frac{dp_x}{dt} + p_x = e^t \qquad \Rightarrow p_x e^t = \frac{1}{2}e^{2t} + c_1$$

$$\dot{p}_y = -\frac{\partial H}{\partial_y} = p_y + y \Rightarrow \frac{dp_y}{dt} - p_y = y = e^{-t} \Rightarrow p_y e^{-t} = -\frac{1}{2}e^{-2t} + c_2$$

Using initial conditions. $\left(p_x(0), p_y(0)\right) = \frac{1}{2}, -\frac{1}{2}$

We get, $c_1 = 0, c_2 = 0$

$\therefore \qquad p_x = \frac{1}{2}e^t$ and $p_y = -\frac{1}{2}e^{-t}$

Multiplying p_x and p_y, we get

$$p_x \cdot p_y = -\frac{1}{4}$$

This is also equation of rectangular hyperbola.

Correct answer : (d)

4. If the Lagrangian of a particle moving in one dimensions is given by $L = \frac{\dot{x}^2}{2x} - V(x)$, the Hamiteonian is:

(a) $\frac{1}{2}xp^2 + V(x)$ (b) $\frac{\dot{x}^2}{2x} + V(x)$ (c) $\frac{1}{2}\dot{x}^2 + V(x)$ (d) $\frac{p^2}{2x} + V(x)$ **[NET June 2012]**

Soln. $L = \frac{\dot{x}^2}{2x} - v(x)$ and $H = p\dot{x} - L$; Now, $p = \frac{\partial L}{\partial \dot{x}} \Rightarrow p = \frac{2\dot{x}}{2x} = \frac{\dot{x}}{x} \Rightarrow \dot{x} = xp$

Therefore,

$$H = p(xp) - L = xp^2 - \left[\frac{(xp)^2}{2x} - V(x)\right] = xp^2 - \frac{x^2p^2}{2x} + V(x) = xp^2 - \frac{1}{2}xp^2 + V(x) = \frac{xp^2}{2} + V(x)$$

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Correct answer is (a)

5. The Hamiltonian of a simple pendulum consisting of a mass 'm' attached to a massless string of length l is $H = \frac{p_\theta^2}{2m\ell^2} + mg\ell(1-\cos\theta)$. If L denotes the Lagrangian, the value of $\frac{dL}{dt}$ is :

(a) $-\frac{2g}{\ell}p_\theta \sin\theta$ (b) $-\frac{g}{\ell}p_\theta \sin 2\theta$ (c) $\frac{g}{\ell}p_\theta \cos\theta$ (d) $\ell\, p_\theta^2 \cos\theta$

Soln. $H = \frac{p_\theta^2}{2m\ell^2} + mg\ell(1-\cos\theta) \Rightarrow L = \sum_i p_i\dot{q}_i - H = p_\theta\dot{\theta} - H$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2} \text{ and } \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mg\ell\sin\theta$$

Therefore, $L = p_\theta\frac{p_\theta}{m\ell^2} - \left[\frac{p_\theta^2}{2m\ell^2} + mg\ell(1-\cos\theta)\right] \Rightarrow L = \frac{p_\theta^2}{2m\ell^2} - mg\ell(1-\cos\theta)$

Since, $L = L(\theta, p_\theta)$ then $\frac{dL}{dt} = \frac{\partial L}{\partial p_\theta}\dot{p}_\theta + \frac{\partial L}{\partial \theta}\dot{\theta}$

Now, $\frac{\partial L}{\partial p_\theta} = \frac{p_\theta}{m\ell^2}$, $\dot{p}_\theta = -mg\ell\sin\theta$, $\frac{\partial L}{\partial \theta} = -mg\ell\sin\theta$, $\dot{\theta} = \frac{p_\theta}{m\ell^2}$

Then $\frac{dL}{dt} = \frac{p_\theta}{m\ell^2}(-mg\ell\sin\theta) - mg\ell\sin\theta\frac{p_\theta}{m\ell^2} = -\frac{2g}{\ell}p_\theta\sin\theta$

Correct option is (a)

6. The particel of mass m is constrained to move on the surface of a cylinder of radius a under an attractive central force F, given by

$$F = -kr \qquad \text{... (i)}$$

where k is the force constant. This force is proportional to the distance of the particle from the origin

The motion of the particle can be described in terms of the cartesian coordinates x, y, z or cylindrical ρ, θ, z (See figure). The equation of constraint is

$$\rho^2 = x^2 + y^2 = a^2$$

To find velocity in cylinderical coordinates, we can proceed as follows

The change in position along $\rho = d\rho$

The change in position along $\theta = \rho d\theta$

The change in position along $z = dz$

So, $v_\rho = \dot{\rho}$, $v_\theta = \rho\dot{\theta}$ and $v_z = \dot{z}$

So, we have $v^2 = v_\rho^2 + v_\theta^2 + v_z^2$

The kinetic energy, $T = \frac{1}{2}mv^2 = \frac{1}{2}\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right)$

Here, $\rho = a$

Therefore, $\dot{\rho} = 0$, and hence,

$$T = \frac{1}{2}m\left(a^2\dot{\theta}^2 + \dot{z}^2\right)$$

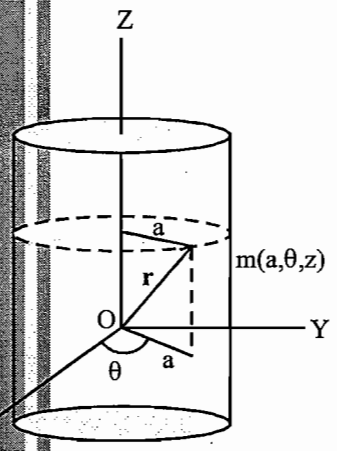

Figure: Motion of a particle on the surface of a cylinder

and $$V = \frac{1}{2}kr^2 = \frac{1}{2}k\left(x^2 + y^2 + z^2\right) = \frac{1}{2}k\left(a^2 + z^2\right)$$

$\therefore$ $$L = T - V = \frac{1}{2}m\left(a^2\dot{\theta}^2 + \dot{z}^2\right) - \frac{1}{2}k\left(a^2 + z^2\right) \qquad \text{... (ii)}$$

Hence, $$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta} \text{ or } \dot{\theta} = \frac{p_\theta}{ma^2}$$

and $$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \text{ or } \dot{z} = \frac{p_z}{m}$$

Therefore, $$H = T + V = \frac{1}{2}m\left(a^2\dot{\theta} + \dot{z}^2\right) + \frac{1}{2}k\left(a^2 + z^2\right)$$

Or, $$H = \frac{p_\theta^2}{2ma^2} + \frac{p_z^2}{m} + \frac{1}{2}k\left(a^2 + z^2\right) \qquad \text{... (iii)}$$

Hence, the Hamilton's equation are

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$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \quad \text{or} \quad p_z = m\dot{z} \qquad \text{... (iv)}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ma^2} \quad \text{or} \quad p_\theta = ma^2\dot{\theta} \qquad \text{... (v)}$$

$$-\dot{p}_z = \frac{\partial H}{\partial z} = kz \quad \text{or} \quad \dot{p}_z = -kz \qquad \text{... (vi)}$$

$$-\dot{p}_\theta = \frac{\partial H}{\partial \theta} = 0 \quad \text{or} \quad p_\theta = \text{constant} \qquad \text{... (vii)}$$

From equation (iv) and (vi), we get

$$m\ddot{z} + kz = 0$$

which shows that the motion of the particle in z direction is simple harmonic with period T, given by

$$T = 2\pi\sqrt{\frac{m}{k}}$$

From equation (v) and (vii), we get

$$p_\theta = ma^2\dot{\theta} = \text{constant}$$

Thus the angular momentum about Z-axis is a constant of motion.

Canonical Transformation:

(a) Definition: There are a number of problems in mechanics for the solution of which it is often desired to change from one set of position and momentum coordinates to a new set of position and momentum coordinates which may be more suitable. Suppose $q_1, q_2, q_3, \ldots\ldots q_n$ and $p_1, p_2, p_3, \ldots\ldots\ldots, p_n$ are one set of coordinates and $Q_1, Q_2, \ldots\ldots Q_n$ and $P_1, P_2, P_3, \ldots\ldots P_n$ are second set of coordinates such that

$$Q_i = Q_i(q_1, q_2, \ldots\ldots\ldots q_n, p_1, p_2 \ldots\ldots p_n)$$

$$P_i = P_i(q_1, q_2, \ldots\ldots q_n, p_1, p_2, \ldots\ldots p_n)$$

then the transformation is said to be canonical or contact (meaning regularised or standard) if the Hamiltonian equation of motion in the new set retains its form, that is, if H is the new hamiltonian then

$$\dot{P}_i = -\frac{\partial H}{\partial Q_i} \text{ and } \dot{Q}_i = \frac{\partial H}{\partial P_i}$$

hold good.

(i) Generating function and condition for canonical transformation : The basic condition for transformation to be canonical is that the form of Hamiltonian is not changed. This is possible when the Hamiltonian's principle also holds for the two Lagranges, that is,

We must have $\delta\int_{t_1}^{t_2} L\,dt = 0$ and $\delta\int_{t_1}^{t_2} \bar{L}\,dt = 0$

Now, $H = \Sigma p_i\dot{q}_i - L$ and $\bar{H} = \Sigma P_i\dot{Q}_i - \bar{L}$

Or, $L = \Sigma p_i\dot{q}_i - H$ and $\bar{L} = \Sigma P_i\dot{Q}_i - \bar{H}$

$\therefore$ $\delta\int_{t_1}^{t_2}\left(\Sigma p_i\dot{q}_i - H\right)dt = 0$ and $\delta\int_{t_1}^{t_2}\left(\Sigma P_i\dot{Q}_i - \bar{H}\right)dt = 0$

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These two equations are simultaneously valid if the integrals of them differ by a total time derivative of an arbitrary function G, that is,

$$\left(\Sigma p_i \dot{q}_i - H\right) - \left(\Sigma P_i \dot{Q}_i - \bar{H}\right) = \frac{dG}{dt}$$

Therefore, canonical transformation needs

$$\delta \int_{t_1}^{t_2} \left(\Sigma p_i \dot{q}_i - H\right) dt - \delta \int_{t_1}^{t_2} \left(\Sigma P_i \dot{Q}_i - \bar{H}\right) dt = \delta \int_{t_1}^{t_2} \frac{dG}{dt} dt$$

Or, $$\delta \int_{t_1}^{t_2} \frac{dG}{dt} dt = 0 \text{ or } \delta\left[G(t_2) - G(t_1)\right] = 0$$

This function G is called the 'generating function'. This is so called because if this function is given or kown in terms of any two of the four variables p, q, P and Q (here each one is representative of the whole set, that is , p stands for $p_1, p_2, p_2 \ldots\ldots p_n$, q stands for $q_1, q_2, \ldots\ldots q_n$ and so on), the remaining coordinates can be estimated. This means that this function generates the unknown coordinates. Since there are four variables (p,q) and (P,Q), they can be grouped in four ways $(Q,q),(q,P),(P,p),(p,Q)$ (taken in order from the corners of rectangle $Qp\,Pq$), there may be four generating functions G_1, G_2, G_3 and G_4 respectively.

To prove the point let us take up first pair and assume that $G = G_1\left(q_1, q_2, \ldots\ldots q_n, Q_1, Q_2, \ldots\ldots Q_n, t\right)$

Then, $$\frac{dG_1}{dt} = \Sigma\left(\frac{dG_1}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial G_1}{\partial Q_i}\frac{dQ_i}{dt}\right) + \frac{\partial G_1}{dt}$$

By definition of generating function,

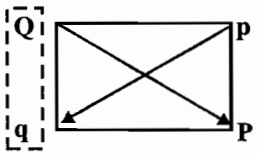
$$\frac{dG_1}{dt} = \Sigma\left(p_i \dot{q}_i - H\right) - \Sigma\left(P_i \dot{Q}_i - \bar{H}\right)$$

$$\therefore \quad \left(\Sigma p_i \dot{q}_i - H\right) - \left(\Sigma P_i \dot{Q}_i - \bar{H}\right) = \Sigma\left(\frac{\partial G_1}{\partial q_i}\dot{q}_i + \frac{\partial G_1}{\partial Q_i}\dot{Q}_i\right) + \frac{\partial G_1}{\partial t}$$

Or, $$\Sigma\left(p_i - \frac{\partial G_1}{\partial q_i}\right)\dot{q}_i - \Sigma\left(P_i + \frac{\partial G_1}{\partial Q_i}\right)\dot{Q}_i + \bar{H} - H - \frac{\partial G_1}{\partial t} = 0$$

Or, $$\Sigma\left(P_i - \frac{\partial G_1}{\partial q_i}\right)dq_i - \Sigma\left(P_i + \frac{\partial G_1}{\partial Q_i}\right)dQ_i + \left(\bar{H} - H - \frac{\partial G_1}{\partial t}\right)dt = 0$$

As q_i's and Q_i's and t are independent variables, the coefficients of dq_i and dQ_i and dt must be separately zero.



$$\left.\begin{aligned} \therefore \quad p_i &= \frac{dG_1}{dq_i} \\ P_i &= -\frac{\partial G_1}{\partial Q_i} \\ \text{and} \quad \bar{H} - H &= \frac{\partial G_1}{\partial t} \end{aligned}\right\}$$

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Let us suppose now that the generating function is a function q and P, that is, $G = G_2(q_1, q_2, \ldots\ldots q_n, P_1, P_2, \ldots\ldots P_n, t)$. Then by the well known Legendre transformation rule. (This rule states that if f_1 and f_2 are two function is equal to the first function plus the summation of the product of uncommon variables),

$$G_2(q_i, P_i, t) = G_1(q_i, Q_i, t) + \Sigma P_i Q_i \quad \text{or,} \quad G_1(q_i, Q_i, t) = G_2(q_i, P_i, t) + \Sigma P_i Q_i$$

Substituting this in the formula introducing G, that is,

$$\frac{dG}{dt} = \left(\Sigma p_i\, \dot{q}_i - H\right) - \left(\Sigma P_i \dot{Q}_i - \bar{H}\right), \text{ we have}$$

$$\left(\Sigma p_i \dot{q}_i - H\right) - \left(\Sigma P_i \dot{Q}_i - \bar{H}\right) = \frac{dG_1}{dt} = \frac{dG_2}{dt} - \frac{d}{dt}\Sigma P_i Q_i$$

$$= \Sigma\left(\frac{\partial G_2}{\partial q_i}\dot{q}_i + \frac{\partial G_2}{\partial P_i}\dot{P}_i\right) + \frac{\partial G_2}{\partial t} - \Sigma\left(P_i \dot{Q}_i + \dot{P}_i Q_i\right)$$

Or,
$$\Sigma p_i \dot{q}_i - H + \bar{H} = \Sigma \frac{\partial G_2}{\partial q_i}\dot{q}_i + \Sigma \frac{\partial G_2}{\partial P_i}\dot{P}_i + \frac{\partial G_2}{\partial t} - \Sigma \dot{P}_i Q_i$$

Or,
$$\Sigma\left(p_i - \frac{\partial G_2}{\partial q_i}\right)\dot{q}_i - H + \bar{H} = \Sigma\left(\frac{\partial G_2}{\partial P_i} - Q_i\right)\dot{P}_i + \frac{\partial G_2}{\partial t}$$

Or,
$$\Sigma\left(p_i - \frac{\partial G_2}{\partial q_i}\right)dq_i + \left(\bar{H} - H\right) = \Sigma\left(\frac{\partial G_2}{\partial P_i} - Q_i\right)dP_i - \frac{\partial G_2}{\partial t}dt$$

Since q_i's and P_i's and t are independent variables, the coefficients of dq_i, dP_i and dt must vanishes separately.

$\therefore$
$$\dot{P}_i = \frac{\partial G_2}{\partial q_i}$$

$$Q_i = \frac{\partial G_2}{\partial P_i}$$

$$\bar{H} - H = \frac{\partial G_2}{\partial t}$$

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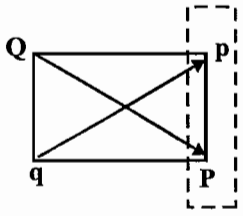
Similarly,
$$q_i = -\frac{\partial G_3}{\partial p_i}$$

$$Q_i = \frac{\partial G_3}{\partial P_i}$$

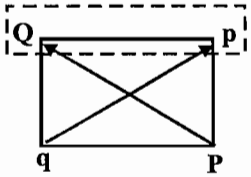
$$\bar{H} - H = \frac{\partial G_3}{\partial t};$$



$$q_i = -\frac{\partial G_4}{\partial p_i}$$

$$P_i = -\frac{\partial G_4}{\partial Q_i}$$

$$\bar{H} - G = \frac{\partial G_4}{\partial t}$$



PROBLEMS

1. Show that for a transformation to be canonical function $\Sigma\left(p_i dq_i - P_i dQ_i\right)$ must be a perfect differential if the generating function does not contain time.

Proof : We have by the property of generating function in terms of q, Q that is, $G = G\left(q_i, Q_i, t\right)$

$$p_i = \frac{\partial G}{\partial q_i},\ P = -\frac{\partial G}{\partial Q_i} \text{ and } \bar{H} - H = \frac{\partial G}{\partial t}$$

Now, $$dG = \Sigma\left(\frac{\partial G}{\partial q_i} dq_i + \frac{\partial G}{\partial Q_i} dQ_i\right) + \frac{\partial G}{\partial t} dt = \Sigma\left(p_i dq_i - P_i dQ_i\right) + 0 \qquad \left(\because \frac{\partial G}{\partial t} = 0\right)$$

Hence, $\Sigma\left(p_i dq_i - P_i dQ_i\right)$ is a perfect differential.

2. Show that $P = \frac{1}{2}\left(p^2 + q^2\right),\ Q = \tan^{-1}\frac{q}{p}$ is canonical.

Proof : Since the transformation equations do not contain time, the generating function also does not contain time. Hence, the condition for transformation to be canonical is that $\left(pdq - PdQ\right)$ be a perfect differential. Here we consider only one term in the summation because the system for which transformation equations are given is of one degree of freedom.

Now, $$pdq - PdQ = pdq - \frac{p^2 + q^2}{2} d\left(\tan^{-1}\frac{q}{p}\right)$$

$$= pdq - \frac{p^2 + q^2}{2} \frac{pd - qdp}{p^2 + q^2} \qquad \left(\because d\left(\tan^{-1} x\right) = \frac{dx}{1 + x^2}\right)$$

$$= pdq - \frac{1}{2} pdq + \frac{1}{2} qdp = \frac{1}{2} pdq + \frac{1}{2} qdp$$

$$= d\left(\frac{1}{2} pq\right)$$

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3. Find the values of α and β so that $Q = q^\alpha \cos p\beta$ and $P = q^\alpha \sin p\beta$ may be canonical. Also find the generating function, $G = G\left(p, Q\right)$ is perfect differential.

Soln. Here the transformation equations do not contain time. Hence the condition for canonical transformation is that $pdq - PdQ$.

Here, $$pdq - PdQ = pdq - q^\alpha \sin p d\left(q^\alpha \cos p\beta\right)$$

$$= pdq - q^\alpha \sin p\,\beta\left(\alpha q^{\alpha-1} \cos p\beta - \beta q^\alpha \sin p\beta dp\right)$$

$$= \left(p - \alpha q^{2\alpha-1} \sin p\beta \cos p\beta\right) dq + \beta q^{2\alpha} \sin^2 p\beta dp$$

Now, $M\left(x, y\right) dx + N\left(x, y\right) dy$ is an exact differential when $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. By this principle $pdq - PdQ$ is a perfect differential if

$$\frac{\partial}{\partial p}\left(p - \alpha q^{2\alpha-1} \sin p\beta \cos p\beta\right) = \frac{\partial}{\partial q}\left(\beta q^{2\alpha} \sin^2 p\beta\right)$$

Or, $1-\alpha\beta q^{2\alpha-1}\left(1-2\sin^2 p\beta\right)=2\alpha\beta q^{2\alpha-1}\sin^2 p\beta$

Or, $1-\alpha\beta q^{2\alpha-1}+2\alpha\beta q^{2\alpha-1}\sin^2 p=2\alpha\beta q^{2\alpha-1}\sin^2 p\beta$

Or, $\alpha\beta\, q^{2\alpha-1}=1$

Since, RHS is a constant, this equation is satisfied only when the power of q is zero. Hence, $2\alpha-1=0$ or $\alpha=\frac{1}{2}$ and $\beta=2$. Thus, $\alpha=\frac{1}{2}$ and $\beta=2$ are the required values of α and β

Generating function : When the generating function is assumed to be $G=G(p,Q)$, we have

$$q=-\frac{\partial G}{\partial p} \text{ and } P=-\frac{\partial G}{\partial Q}$$

Here, $Q=q^{1/2}\cos 2p$ or $q=\left(\frac{Q}{\cos 2p}\right)^2$

And $P=q^{1/2}\sin 2p=\left(\frac{Q}{\cos 2p}\right)\sin 2p=Q\tan 2p$

$\therefore$ $\frac{\partial G}{\partial p}=-q=-\left(\frac{Q}{\cos 2p}\right)^2$ or $G=-Q^2\int\frac{\partial p}{\cos^2 2p}$

Or, $G=-Q^2\int\sec^2 2p\partial p=-\frac{1}{2}Q^2\tan 2p$

Thus, $G=-\frac{1}{2}Q^2\tan 2p$ is the required generating function.

4. Show that the generating function for the transformation $p=\frac{1}{Q}$ and $q=PQ^2$ is $G=\frac{q}{Q}$

Proof : Since we have to show that $G=\frac{q}{Q}$ we assume that the generating function needed is a function of q, Q that is, $G=G(q,Q)$. When q and Q are independent variables, the other two variables p and P are given by $p=\frac{\partial G}{\partial q}$ and $P=-\frac{\partial G}{\partial Q}$.

Here, $p=\frac{1}{Q}$ and hence here $\frac{1}{Q}=\frac{\partial G}{\partial q}$ or $G=\int\frac{\partial q}{Q}$.

Or $G=\frac{q}{Q}$

From the second relation also we can show that

$$G=\frac{q}{Q}$$

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5. Show that if there exists a generating function G such that $\frac{dG}{dt} = L - \bar{L}$ where L and $\bar{L}$ are the Lagrangians in old and new set of coordinates, then the transformation is canonical.

Proof : By Hamilton's principle, $\frac{dG}{dt} = L - \delta\int_{t_1}^{t_2} L dt = 0$ and $\delta\int_{t_1}^{t_2} \bar{L} dt = 0$.

Subtracting we have, $\delta\int_{t_1}^{t_2}\left(L - \bar{L}\right)dt = 0$

By virtue of the given relation i.e. $\frac{dG}{dt} = L - \bar{L}$

We have, $\delta\int_{t_1}^{t_2}\frac{dG}{dt}dt = 0$, or $\delta\left[G(t_2) - G(t_1)\right] = 0$ which is the condition for canonical transformation.

Hence under the given condition the transformation is canonical.

6. Show that $G = -\Sigma Q_i p_i$ generating on identity transformation.

Proof : Let us define an identity transformation. When the coordinates of old set are equal to the coordinates of the new set, it is called an identity transformation.

It is given here that the generating function is a function of p, Q i.e. $G = G(p, Q)$

In this case we have, $q_i = -\frac{\partial G}{\partial p_i}$ and $P_i = -\frac{\partial G}{\partial Q_i}$

Here it is given that $G = -\Sigma p_i Q_i$. As we have to take derivative of G with respect to p_i, we change the suffix in the summation to j i.e. we take $G = -\Sigma p_j Q_j$.

$$\therefore \quad \frac{\partial G}{\partial p_i} = -\Sigma\left(p_j\frac{\partial Q_j}{\partial p_i} + Q_j\frac{\partial p_j}{\partial p_i}\right)$$

Now, $\frac{\partial Q_j}{\partial p_i} = 0$ because, p, Q are selected as independent variables

here and $\frac{\partial p_j}{\partial p_i} = 0$ if $i \neq j = 1$ if $i = j$.

Here, $\frac{\partial G}{\partial p_i} = -Q_i$. But $\frac{\partial G}{\partial p_i} = -q_i$ (in general).

$$\therefore \quad q_i = Q_i$$

$$\frac{\partial G}{\partial Q_i} = -\Sigma\left(Q_j\frac{\partial p_j}{\partial Q_i} + p_j\frac{\partial Q_j}{\partial Q_i}\right) = -p_i$$

(Because $\frac{\partial p_j}{\partial Q_i} = 0$ and $\frac{\partial Q_j}{\partial Q_i} = 1$ where i = j, otherwise zero).

$$\therefore \quad p_i = P_i$$

CAREER ENDEAVOUR

Brackets : Poisson Brackets:

Brackets, in classical mechanics, are mathematical artifices defined to derive maximum economy of expression, clarity and have easily repreducible form of equation of motion. Two brackets have been defined–one by Poisson known as Poisson brackets and the other by Lagrange known as Lagrange brackets. The latter are of only historical importance.

Definition of Poisson brackets and its properties:

Defintion: Poisson brackets of two functions u and v is defined as

$$[u,v]_{qp}=\sum\left(\frac{\partial u}{\partial q_k}\cdot\frac{\partial v}{\partial p_k}-\frac{\partial u}{\partial p_k}\cdot\frac{\partial v}{\partial q_k}\right)$$

Since summation is always to be there it is customery to omit it but keep it in mind that it is present in all expressions. Hence, Poisson bracket may be written as

$$[u,v]_{p,q}=\frac{\partial u}{\partial q_k}\cdot\frac{\partial v}{\partial q_k}-\frac{\partial u}{\partial p_k}\cdot\frac{\partial v}{\partial q_k}$$

when u and v are from among p's and q's, Poisson brackets are specially called fundamental brackets.

Propeties : (i) Poisson bracket of a function with itself is identically zero.

Proof : From definition it follows that

$$[u,u]=\left(\frac{\partial u}{\partial q_k}\cdot\frac{\partial u}{\partial p_k}-\frac{\partial u}{\partial p_k}\cdot\frac{\partial u}{\partial q_k}\right)=0$$

(ii) The Poisson bracket of a function with a constant is zero.

Proof : From definition it follows

$$[u,c]=\left(\frac{\partial u}{\partial q_k}\cdot\frac{\partial c}{\partial p_k}-\frac{\partial u}{\partial p_k}\cdot\frac{\partial c}{\partial q_k}\right)=0$$

(iii) Poisson bracket obeys distribution law, that is,

$$[u+v,\varphi]=[u,\varphi]+[v,\varphi]$$

Proof : By defintion, $[u+v,\varphi]=\left(\frac{\partial(u+v)}{\partial q_k}\cdot\frac{\partial\varphi}{\partial q_k}-\frac{\partial(u+v)}{\partial p_k}\cdot\frac{\partial\varphi}{\partial q_k}\right)$

Or, $$[u+v,\varphi]=\left(\frac{\partial u}{\partial q_k}\cdot\frac{\partial\varphi}{\partial p_k}-\frac{\partial u}{\partial p_k}\cdot\frac{\partial\varphi}{\partial q_k}\right)+\left(\frac{\partial v}{\partial q_k}\cdot\frac{\partial\varphi}{\partial p_k}-\frac{\partial v}{\partial p_k}\cdot\frac{\partial\varphi}{\partial q_k}\right)$$

$$=[u,\varphi]+[v,\varphi]$$

(iv) The Poisson brackets do not obey commutative law, that is, $[u,v]\neq[v,u]$ rather $[u,v]=-[v,u]$

Proof : $[u,v]=\left(\frac{\partial u}{\partial q_k}\cdot\frac{\partial v}{\partial p_k}-\frac{\partial u}{\partial p_k}\cdot\frac{\partial v}{\partial q_k}\right)=-\left(\frac{\partial v}{\partial q_k}\cdot\frac{\partial u}{\partial p_k}-\frac{\partial v}{\partial q_k}\cdot\frac{\partial u}{\partial q_k}\right)=-[v,u]$

(v) The Poisson bracket of product of two functions and an arbitrary function is equal to the summation of Poisson brackets of the functions with that arbitrary function multiplied cross-wise by the functions, that is, $[uv,\varphi]=u[v,\varphi]+v[u,\varphi]$ that is, a rule similar to product rule of differentiation.

Proof : $$[uv,\varphi]=\left(\frac{\partial(uv)}{\partial q_k}\cdot\frac{\partial\varphi}{\partial p_k}-\frac{\partial(uv)}{\partial p_k}\cdot\frac{\partial\varphi}{\partial q_k}\right)$$

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$$= \left(u\frac{\partial v}{\partial q_k} + v\frac{\partial u}{\partial q_k} \right)\frac{\partial \varphi}{\partial p_k} - \left(u\frac{\partial v}{\partial p_k} + v\frac{\partial u}{\partial p_k} \right)\frac{\partial \varphi}{\partial q_k}$$

$$= u\left(\frac{\partial v}{\partial q_k}\cdot\frac{\partial \varphi}{\partial p_k} - \frac{\partial v}{\partial p_k}\cdot\frac{\partial \varphi}{\partial q_k} \right) + v\left(\frac{\partial u}{\partial q_k}\cdot\frac{\partial \varphi}{\partial p_k} - \frac{\partial u}{\partial p_k}\cdot\frac{\partial \varphi}{\partial p_k} \right)$$

$$= u[v,\varphi] + v[u,\varphi]$$

(vi) Poisson bracket of any function with one of generalised coordinates is the partial derivative of that function with the corresponding generalised momentum with negative sign and vice-versa but with change of sign, that is, $[u,q_i] = -\frac{\partial u}{\partial p_i}$ and $[u,p_i] = \frac{\partial u}{\partial q_i}$. It can be proved directly from the definition.

(vii) Poisson bracket of generalised coordinate with generalised momentum is Kronecker delta, that is,

$$[q_i, p_j] = \delta_{ij} \text{ where } \delta_{ij} = 1 \text{ if } i = j \text{ and} = 0 \text{ if } i \neq j$$

(viii) **Property of Poisson Double Brackets:**

If u, v, w are three functions then the bracket formed by the bracket of any two with the third is called Poisson Doubler Bracket i.e. $[[u,v],w]$, $[[v,w],u]$, $[[w,u],v]$ are the three Poisson Double Brackets, when u, v and w are taken in cyclic order.

The sum of these three brackets is always zero. This is called Jacobi Identity for Poisson Brackets.

$$[[u,v],w] + [[v,w],u] + [[w,u],v] = 0$$

And $$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$$

(ix) Poisson Brackets remain invariant under canonical transformation.

Proof : We have to show $[u,v]_{q,p} = [u,v]_{Q,P}$

By definition of Poisson Bracket,

$$[u,v]_{q,p} = \frac{\partial u}{\partial Q_k}\cdot\frac{\partial v}{\partial P_k} - \frac{\partial u}{\partial P_k}\cdot\frac{\partial v}{\partial Q_k}$$, omitting the summation symbol which is supposed to be there always.

Since, $v = v(q,p)$, we have by the rule of partial differentiation

$$[u,v]_{Q,P} = \frac{\partial u}{\partial Q_k}\left(\frac{\partial v}{\partial q_j}\frac{\partial q_j}{dP_k} + \frac{\partial v}{\partial p_j}\frac{\partial p_j}{\partial P_k} \right) - \frac{\partial u}{\partial P_k}\left(\frac{\partial v}{\partial q_j}\frac{\partial q_j}{\partial Q_k} + \frac{\partial v}{\partial p_j}\frac{\partial p_j}{\partial Q_k} \right)$$

$$= \frac{\partial v}{\partial q_j}\left(\frac{\partial u}{\partial Q_k}\cdot\frac{\partial q_j}{\partial P_k} - \frac{\partial u}{\partial P_k}\cdot\frac{\partial q_j}{\partial Q_k} \right) + \frac{\partial v}{\partial p_j}\left(\frac{\partial u}{\partial Q_k}\cdot\frac{\partial p_j}{\partial P_k} - \frac{\partial u}{\partial P_k}\cdot\frac{\partial p_j}{\partial Q_k} \right)$$

Or, $$[u,v]_{Q,P} = \frac{\partial v}{\partial q_j}[u,q_j]_{Q,P} + \frac{\partial v}{\partial p_j}[u,p_j]_{Q,P} \qquad \ldots (i)$$

Now, $$[u,q_j]_{Q,P} = -[q_j,u]_{Q,P} = -\left(\frac{\partial q_j}{\partial Q_\ell}\cdot\frac{\partial u}{\partial P_\ell} - \frac{\partial q_j}{\partial P_\ell}\cdot\frac{\partial u}{\partial Q_\ell} \right)$$

Since $u = u(q,p)$, we have by the rule of partial differentiation,

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$$[u,q_j]_{Q,P} = -[q_j,u]_{Q,P} = -\frac{\partial q_j}{\partial Q_j}\left(\frac{\partial u}{\partial q_m}\cdot\frac{\partial q_m}{P_\ell}+\frac{\partial u}{\partial p_m}\frac{\partial p_m}{\partial P_\ell}\right)+\frac{\partial q_j}{\partial P_\ell}\left(\frac{\partial u}{\partial q_m}\cdot\frac{\partial q_m}{\partial Q_\ell}+\frac{\partial u}{\partial p_m}\cdot\frac{\partial p_m}{\partial Q_\ell}\right)$$

Or,
$$[u,q_j]_{Q,P} = -\frac{\partial u}{\partial q_m}\left(\frac{\partial q_j}{\partial Q_\ell}\cdot\frac{\partial q_m}{\partial P_\ell}-\frac{\partial q_j}{\partial P_\ell}\cdot\frac{\partial q_m}{\partial Q_\ell}\right)+\frac{\partial u}{\partial p_m}\left(\frac{\partial p_f}{\partial Q_\ell}\cdot\frac{\partial p_m}{\partial P_\ell}-\frac{\partial p_m}{\partial P_\ell}\cdot\frac{\partial p_m}{\partial Q_\ell}\right)$$

$$= -\frac{\partial u}{\partial q_m}[q_f, q_m]_{Q,P}+\frac{\partial u}{\partial p_m}[q_j, p_m]_{Q,P}$$

$$= 0+\frac{\partial u}{\partial p_m}(-\delta_{jm})$$

(because self Poisson bracket is always zero and Poisson Bracket of q_j and p_m is Kronecker delta)

Or,
$$[u,q_j]_{Q,P} = -\frac{\partial u}{\partial p_j}$$

Similarly,
$$[v,p_j]_{Q,P} = +\frac{\partial u}{\partial q_j}$$

Substituting these results in (i)

$$[u,v]_{Q,P} = \frac{\partial v}{\partial q_f}\left(-\frac{\partial u}{\partial p_j}\right)-\frac{\partial v}{\partial p_j}\cdot\frac{\partial u}{\partial q_j}$$

$$= -\left(\frac{\partial v}{\partial q_j}\cdot\frac{\partial u}{\partial p_j}+\frac{\partial v}{\partial p_j}\cdot\frac{\partial u}{\partial q_j}\right)$$

$$= -[u,v]_{q,p} = [u,v]_{p,q}$$

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Importance of Poisson Brackets.

(i) Hamilton's equation of motion takes a very convenient and easily reproducible form in terms of Poisson Brackets.

We have as Hamilton's equation of motion,

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \text{ and } \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Now by the specific property of Poisson Bracket that Poisson bracket of any function with generalised coordinate is equal to the derivative of the function with the corresponding generalised moment, we have

$$[H,q_i] = -\frac{\partial H}{\partial p_i} \text{ and } [H,p_i] = \frac{\partial H}{\partial q_i}$$

Using these brackets we can write the equations of motion as

$$[H,q_i] = -\dot{q}_i,\ [H,p_i] = -\dot{p}_i$$

Or,
$$[q_i,H] = \dot{q}_i \text{ and } [p_i,H] = \dot{p}_i \qquad (\because [u,v] = -[v,u])$$

Now see how it becomes easy to remember and reproduce Hamilton's equation of motion.

(ii) It helps to seek and identify constants of motion of a system.

All functions whose Poisson Bracket with Hamiltonian vanish is a constant or integral of motion. Conversely, the Poisson bracket of a constant of motion is zero.

Proof : Suppose u is a function of q, p and t, that is, $u = u(q, p, t)$

Then, $$\frac{du}{dt} = \sum\left(\frac{\partial u}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial u}{\partial p_i}\frac{dp_i}{dt}\right) + \frac{\partial u}{\partial t}$$

We have, $$\dot{q}_i = \frac{\partial H}{\partial p} \text{ and } \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$\therefore$ $$\frac{du}{dt} = \sum\left(\frac{\partial u}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial H}{\partial q_i}\right) + \frac{\partial u}{\partial t}$$

$$= [u, H]_{q,p} + \frac{\partial u}{\partial t}$$

If u does not contain t explicitly then $\frac{\partial u}{\partial t} = 0$ (when $u = u(p, q)$, it is an explicit function and when $u(p, q, t) = 0$, it is an implicit function) and $\frac{du}{dt} = [u, H]$.

Further if $[u, H] = 0$, then $\frac{du}{dt} = 0$ or $u = a$ constant. Thus the test of searching a constant of motion not involving time explicitly reduces to requiring their Poisson Bracket with the Hamiltonian vanish.

Example: Show that Poisson Bracket of two integrals of motion is itself a constant of motion. This is known a Poisson Theorem.

Proof : Suppose L_x and L_y be two constants of motion, then their Poisson Bracket with H must be zero that is,

$$[L_x, H] = 0 \text{ and } [L_y, H] = 0$$

Now by jacobi's identity we have

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$$

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Taking L_x, L_y and H in place of u, v and w respectively, we have

$$[L_x,[L_y, H]] + [L_y,[H, L_x]] + [H,[L_x, L_y]] = 0$$

Or, $$[L_x, 0] + [L_y, 0] + [H,[L_x, L_y]] = 0$$

Since, Poisson Bracket of a function with a constant is zero, we have,

$$[H,[L_x, L_y]] = 0$$

$\therefore$ $[L_x, L_y]$ = an integral of motion.

Problem: The Poisson bracket $\{|r|, |p|\}$ has the value **[NET June 2012]**

(a) $|r||p|$ (b) $\hat{r}.\hat{p}$ (c) 3 (d) 1

Soln. $\left\{\left(x^2 + y^2 + z^2\right)^{1/2}, \left(p_x^2 + p_y^2 + p_z^2\right)^{1/2}\right\}$

$$=\frac{\partial\left(x^2+y^2+z^2\right)^{1/2}}{\partial x}\frac{\partial\left(p_x^2+p_y^2+p_z^2\right)^{1/2}}{\partial p_x}+\frac{\partial\left(x^2+y^2+z^2\right)^{1/2}}{\partial y}\frac{\partial\left(p_x^2+p_y^2+p_z^2\right)^{1/2}}{\partial p_y}$$

$$+\frac{\partial\left(x^2+y^2+z^2\right)^{1/2}}{\partial z}\frac{\partial\left(p_x^2+p_y^2+p_z^2\right)^{1/2}}{\partial p_z}$$

$$=\left[\frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}.2x\right]\left[\frac{1}{2}\left(p_x^2+p_y^2+p_z^2\right)^{-1/2}2p_x\right]+$$

$$\left[\frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}.2y\right]\left[\frac{1}{2}\left(p_x^2+p_y^2+p_z^2\right)^{-1/2}2p_y\right]+\left[\frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}.2z\right]\left[\frac{1}{2}\left(p_x^2+p_y^2+p_z^2\right)^{-1/2}2p_z\right]$$

$$=\frac{x}{r}\frac{p_x}{p}+\frac{y}{r}.\frac{p_y}{p}+\frac{z}{r}.\frac{p_z}{p}=\frac{xp_x+yp_y+zp_z}{rp}=\frac{\vec{r}.\vec{p}}{rp}=\hat{r}.\hat{p}$$

Correct answer is (b).

Problem: A system is governed by the Hamiltonian $H=\frac{1}{2}(p_x-ay)^2+\frac{1}{2}(p_y-bx)^2$ **[NET June 2013]**

where a and b are constants and p_x, p_y are momenta conjugate to x and y respectively. For what values of a and b will the quantities $(p_x - 3y)$ and $(p_y + 2x)$ be conserved?

(a) a = –3, b = 2 (b) a = 3, b = –2 (c) a = 2, b = –3 (d) a = –2, b = 3

Soln. $P_x - 3y$ and $P_y + 2x$ are constants of motion.

$\therefore \{p_x - 3y, H\} = 0$

Let $p_x - 3y = \alpha$

$\therefore \{\alpha, H\} = 0$

$$\therefore \frac{\partial\alpha}{\partial x}.\frac{\partial H}{\partial p_x}-\frac{\partial\alpha}{\partial p_x}.\frac{\partial H}{\partial x}+\frac{\partial\alpha}{\partial y}.\frac{\partial H}{\partial p_y}-\frac{\partial\alpha}{\partial p_y}.\frac{\partial H}{\partial y}=0$$

$\Rightarrow 0 + b(p_y - bx) + (-3)(p_y - bx) = 0$

$\Rightarrow (p_y - bx)(b - 3) = 0$

$p_y - bx \neq 0$

$\therefore b = 3$

Similarly using $\{p_y + 2x, H\} = 0$

We get value of *a*.

Correct answer is (d)

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Hamilton-Jacobi Equation:

Deduction: We know that if the generating function is *G*, then the transformed Hamiltonian is given by

$$\bar{H}=H+\frac{\partial G}{\partial t}$$

where, $\bar{H}=\bar{H}(Q,P)$ and $H=H(q,p)$

As usual we have the canonical equations,

$$\dot{q}_i=\frac{\partial H}{\partial p_i} \text{ and } p_i=-\frac{\partial H}{\partial q_i}$$

$$\dot{Q}_i=\frac{\partial \bar{H}}{\partial P_i} \text{ and } \dot{P}_i=-\frac{\partial \bar{H}}{\partial Q_i}$$

Now suppose that the transformation is such that the new variables are constants in time and the transferred Hamiltonian $\bar{H}$ vanish altogether. Then $\dot{Q}_\ell = 0, \dot{P}_\ell = 0$ and $H + \frac{\partial G}{\partial t} = 0$

It is convenient to take G as a function of the old coordinates q_i, the new momenta P_i, and the time t, that is, $G = G(q,P,t)$. Under this scheme $p_i = \frac{\partial G}{\partial q_i}$, $Q_i = \frac{\partial G}{\partial P_i}$, $\bar{H} = H + \frac{\partial G}{\partial t}$ so that the above equation $\left(viz, H + \frac{\partial G}{\partial t} = 0 \right)$ becomes

$$H\left(q_1, q_2, \ldots\ldots\ldots q_n, \frac{\partial G}{\partial q_1}, \frac{\partial G}{\partial q_2}, \ldots\ldots, \frac{\partial G}{\partial q_n}, t \right) + \frac{\partial G}{\partial t} = 0$$

This equation is known as the Hamilton-Jacobi equation. This is a partial differential equation of $(n+1)$ variables $q_1, q_2, \ldots\ldots q_n, t$. It is customery to denote the solution of this equation (Hamilton-Jacobi equation) by S and call it Hamilton's principal function).

General Treatment of Hamilton-Jacobi Technique and Physical Significane of S:

The Hamilton-Jacobi equation has the form of a first order differential equation in $n + 1$ constant of integration because there are $n + 1$ partial derivatives. Hence we may write in a general way,

$$S = S(q_1, q_2, \ldots\ldots q_n, \alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_{n+1}, t)$$

One of the constants of integration, however, is in fact irrevalent to the solution, for it is seen that S itself does not occur in the Hamilton-Jacobi Equatons; only its partial derivatives with respect to q or t are involved. Hence, if S is some solution of the differential equation, then $S + \alpha$, where α is any constant, must also be a solution. One of the $n + 1$ constants of integration must therefore appear only as an additive constant tacked on to S. But an additive constant has no importance in a generating function since only partial derivatives of the generating function occur in the transformation equations. Hence, for our purpose a complete solution can be written as

$$S = S(q_1, q_2, \ldots\ldots, q_n, \alpha_1, \alpha_2, \ldots\ldots\ldots, \alpha_n, t)$$

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where none of the n independent constants is solely additive. Now our original assumption that S is a function of q, P, t, gives $S = S(q,P,t)$. Comparing these two we are therefore at liberty to take n constants of integrations as n new (constant) momenta, that is, $P_i = \alpha_i$. The n transformation equations for p can be written as

$$p_i = \frac{\partial S(q,\alpha,t)}{\partial q_i}$$

At the time $t = t_0$ these constitute n equations relating to the n $\alpha's$ with the initial p and q values thus enabling one to evaluate the constants of integrations in terms of the specific initial conditions of the problem. The other n transformation equations for Q, the new constant coordinates, appear as

$$Q_i = \beta_i = \frac{\partial S(q,\alpha,t)}{\partial P_i}$$

The constants $(\beta's)$ can be similarly obtained from the initial conditions, simply by calculating the value of right hand side of this equation at $t = t_0$ with the known initial values of q_i. This equation can be 'turned inside out' to furnish q_i in terms of α, β and t. That is, we may write $q_i = q_i(\alpha,,t)$ and $p_i = p_i(\alpha, \beta, t)$.

Physical significance of S : The physical significance is furnished by an examination of its total time derivative. We have $S = S(q, P, t)$

$$\frac{dS}{dt} = \sum\left(\frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial P_i}\dot{P}_i\right) + \frac{\partial S}{\partial t} = \sum\frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t}$$

Since, p_i's are constants in time.

Now, $$p_i = \frac{\partial S}{\partial q_i} \text{ and so } \frac{dS}{dt} = \sum p_i\dot{q}_i - H = L$$

Or, $S = \int L dt + a$ constant. Thus the physical significane of S is that it is the time integral of the Lagrangian of the system plus a constant.

Harmonic Oscillator as illustration of Hamilton-Jacobi Technique:

To illustrate the Hamilton-Jacobi technique for solving the motion of mechanical systems we shall work out in detail the simple problem of a one-dimensional harmonic oscillator.

The case of 1-D Simple Harmonic Oscillator:

To illustarte the Hamilton-Jacobi technique for solving the motion of mechanical systems, we shall work out in detail the simple problem of a one-dimensional harmonic oscillator. The Hamiltonian is

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) \equiv E$$

where $$\omega = \sqrt{\frac{k}{m}}$$

k being the force constant. We obtain the Hamilton-Jacobi equations for S by setting p equal to $\frac{\partial S}{\partial q}$ and substituting in the Hamiltonian; the requirement that the new Hamiltonian vanishes becomes

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0$$

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Since the explicit dependence of S on t is present only in the last term, equation can be used to eliminate the time form the Hamilton-Jacobi equation

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + m^2\omega^2 q^2\right] = \alpha$$

The integration constant α is thus to be identified with the total energy E. This can also be recognized directly from equation and the relation

$$\frac{\partial S}{\partial t} + H = 0$$

which then reduces to

$$H = \alpha$$

Equation (9.16) can be integrated immediately to

$$W = \sqrt{2m\alpha}\int dq\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}}$$

so that $$S = \sqrt{2m\alpha}\int dq\sqrt{1-\frac{m\omega^2 q^2}{2\alpha}} - \alpha t$$

$$\beta' = \frac{\partial S}{\partial \alpha} = \sqrt{\frac{m}{2\alpha}}\int \frac{dq}{\sqrt{1-\frac{m\omega^2 q^2}{2\alpha}}} - t$$

which can be integrated without trouble to give

$$t + \beta' = \frac{1}{\omega}\arcsin q\sqrt{\frac{m\omega^2}{2\alpha}}$$

Equation can be immediately "turned inside out" to furnish q as a function of t and the two constants of integration α and $\beta = \beta'\omega$

$$q = \sqrt{\frac{2\alpha}{m\omega^2}}\sin(\omega t + \beta)$$

which is the familiar solution for a harmonic oscillator. Formally, the solution for the momentum comes from the transformation equation, which, using equation, can be written

$$p = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \sqrt{2m\alpha - m^2\omega^2 q^2}$$

In conjunction with the solution for q, equation , this becomes

$$p = \sqrt{2m\alpha\left(1-\sin^2(\omega t+\beta)\right)}$$

Or, $$p = \sqrt{2m\alpha}\cos(\omega t + \beta)$$

Of course, this result checks with the simple identification of p as $m\dot{q}$.

To complete the story, the constants α and β must be connected with the initial conditions q_0 and p_0 at time $t = 0$. By squaring equation and , it is clearly seen that α as the conserved total energy E. Finally, the phase constant β is related to q_0 and p_0 by

$$\tan\beta = m\omega\frac{q_0}{p_0}$$

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The choice $q_0 = 0$ and hence $\beta = 0$ corresponds to starting the motion with the oscillator at its equilibrium position $q = 0$.

Thus, Hamilton's principle function is the generator of a canonical transformation to a new coordinate that measures the phase angle of the oscillation and to a new canonical momentum identified as the total energy.

If the solution of q is substituted into equation , Hamilton's principal function can be written as

$$S = 2\alpha\int\cos^2(\omega t+\beta)\,dt - \alpha t = 2\alpha\int\left(\cos^2(\omega t+\beta) - \frac{1}{2}\right)dt$$

Now, the Lagrangian is

$$L = \frac{1}{2m}\left(p^2 - m^2\omega^2 q^2\right) = \alpha\left(\cos^2(\omega t+\beta) - \sin^2(\omega t+\beta)\right)$$

$$= 2\alpha\left(\cos^2(\omega t+\beta) - \frac{1}{2}\right)$$

so that S is the time integral of the Lagrangian, in agreement with the general relation ($S = \int L dt + \text{constant}$). Note that the identity could not be proved until after solution to the problem had been obtained.

The case of 2-D Simple Harmonic Oscillator:

If we let m be the mass of the oscillating body and k_x and k_y be the spring constants in the x- and y-direction, respectively, the Hamiltonian is

$$E = \frac{1}{2m}\left(p_x^2 + p_y^2 + m^2\omega_x^2 x^2 + m^2\omega_y^2 y^2\right)$$

where, $\omega_x = \sqrt{\frac{k_x}{m}}$ and $\omega_y = \sqrt{\frac{k_y}{m}}$

Since the coordinates and momenta separate into two distinct sets, the principal function can be written as a sum of the characteristic function for each pair. Assuming that we solve the y-functional dependency first, this means

$$S\left(x, y, \alpha, \alpha_y, t\right) = F_x\left(x, \alpha\right) + F_y\left(y, \alpha\right) - \alpha t$$

and the Hamilton-Jacobi equation assumes the form

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + m^2\omega_x^2 x^2 + \left(\frac{\partial W}{\partial y}\right)^2 + m^2\omega_y^2 y^2\right] = \alpha$$

in analogy with previous case. Since the variables are separated, the y-part of the above equation must be equal to a constant, which we call α_y, so,

$$\frac{1}{2m}\left(\frac{\partial W}{\partial y}\right)^2 + \frac{1}{2}m\omega_y^2 y^2 = \alpha_y$$

and we replace the y-term in with α_y from, yielding

$$\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{1}{2}m\omega_x^2 x^2 = \alpha_x$$

where we write $\alpha - \alpha_y = \alpha_x$ showing the symmetry of equation

Eeach equation has a solution analogous to equation, so

$$x = \sqrt{\frac{2\alpha_x}{m\omega_x^2}}\sin\left(\omega_x t + \beta_x\right)$$

$$p_x = \sqrt{2m\alpha_x}\cos\left(\omega_x t + \beta_x\right)$$

$$y = \sqrt{\frac{2\alpha_y}{m\omega_y^2}}\sin\left(\omega_y t + \beta_y\right)$$

$$p_y = \sqrt{2m\alpha_y}\cos\left(\omega_y t + \beta_y\right)$$

where the β_i's are phase constants and the total energy is given by

$$E = \alpha_x + \alpha_y = \alpha$$

As a third example of Hamilton-Jacobi theory, we again consider the two-dimensional harmonic oscillator, only we will assume the oscillator is isotorpic, so

$$k_x = k_y = k \quad \text{and} \quad \omega_x = \omega_y = \omega$$

and use polar coordinates to write

$$x = r\cos\theta, \qquad r = \sqrt{x^2 + y^2}$$

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$$t = r\sin\theta, \qquad \theta = \tan^{-1}\frac{y}{x}$$

$$p_x = m\dot{x} \qquad p_r = m\dot{r}$$

$$p_y = m\dot{y}, \qquad p_\theta = mr^2\dot{\theta}$$

The Hamiltonian now written as

$$E = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + m^2\omega^2 r^2\right)$$

is cyclic in the angular coordinate θ. The principal function can then be written as

$$S(r,\theta,\alpha,\alpha_\theta) = W_r(r,\alpha) + W_\theta(\theta,\alpha_\theta) - \alpha t$$

$$= W_r(r,\alpha) + \theta\alpha_\theta - \alpha t$$

where, as we show later, a cyclic coordinate q_i always has the characteristic function component $W_{q_i} = q_i\alpha_i$.

The canonical momentum p_θ associated with the cyclic coordinate θ, is calculated from the generating function,

$$p_\theta = \frac{\partial F_\theta}{\partial\theta} = \alpha_\theta$$

and has its expected constant value.

When this p_θ is substituted into equation , $W_r(r,\alpha)$ satisfies

$$\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{2mr^2} + \frac{1}{2}m\omega^2 r^2 = \alpha$$

Rather than solving this equation directly for W_r, we shall write the Cartesian coordinate solution for these conditions as

$$x = \sqrt{\frac{2\alpha}{m\omega^2}}\sin(\omega t + \beta) \qquad p_x = \sqrt{2m\alpha}\cos(\omega t + \beta)$$

$$y = \sqrt{\frac{2\alpha}{m\omega^2}}\sin\omega t \qquad p_y = \sqrt{2m\alpha}\cos\omega t$$

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and use these to get the polar counterparts,

$$r = \sqrt{\frac{2\alpha}{m\omega^2}}\sqrt{\sin^2\omega t + \sin^2(\omega t + \beta)}, \quad p_r = m\dot{r}$$

and

$$\theta = \tan^{-1}\left[\frac{\sin\omega t}{\sin(\omega t + \beta)}\right], \qquad p_\theta = mr^2\dot{\theta}$$

There are two limiting cases. The linear case is when $\beta = 0$, for which

$$r = \sqrt{\frac{4\alpha}{m\omega^2}}\sin\omega t, \qquad p_r = \sqrt{2m\alpha}\cos\omega t$$

and

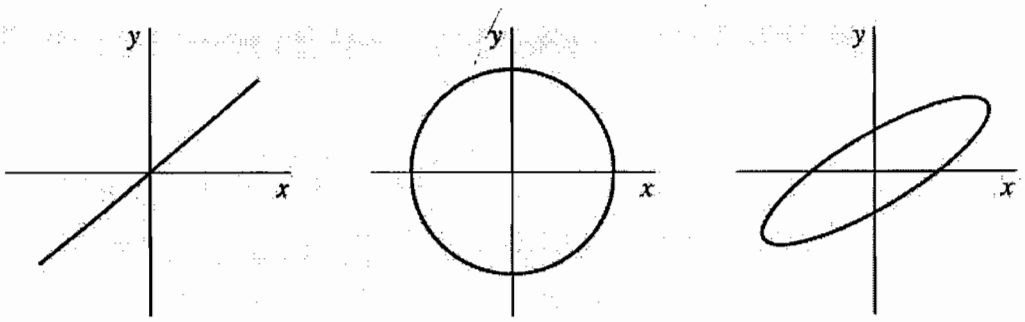
$$\theta = \frac{\pi}{4}, \qquad p_\theta = 0$$

The motion in an x-y plot will be an oscillation along a diagonal line as shown in figure. The other limiting case is when $\beta = \frac{\pi}{2}$, for which

$$r = r_0 = \sqrt{\frac{2\alpha}{m\omega^2}}, \qquad p_r = 0$$

$$\theta = \omega t, \qquad p_\theta = mr_0^2\omega$$

The motion in an *x-y* plot for this limiting case is a circle of radius r_0 as is shown in figure below. For other values of $\beta\left(0 < \beta < \frac{\pi}{2}\right)$, the orbit in coordinate space is an ellipse. The case for $\beta = \frac{\pi}{4}$ is shown in figure below. The plots shown in figure are further examples of Lissajous figures.



(a) $\beta = 0$ (b) $\beta = \frac{\pi}{2}$ (c) $\beta = \frac{\pi}{4}$

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Action-Angle Variables:

This is a versatile technique used for finding all the periodictities in a periodic system without completely solving the HJ equations, and was the first devised by Delaunay in 1846 in connection with some problems in celestial mechanics. However, the name action-angle variables was suggested by Karl Schwarzschild in 1916.

Periodic systems are those which repeat their state of motion after a constant interval of time known as the period. In the phase space all such systems need not retrace their paths. Those which retrace their in phase space are called librating systems. On the othe hand, for the rotating systems some momenta are periodic functions of their conjugate coordinates, but the latter do not return to their original value and increase or decrease monotonically with time. Thus libration corresponds to periodic systems having velocities $\dot{q}_i$ changing sign over any single period, otherwise q_i cannot return to their old values after completion of the period. Rotation leads to a situation where q_i changes monotonically or remains constant while the system returns, after each period in q_i, to its original configuration.

Action variables are useful parameters for systems which are periodic, conservative and orthogonally decomposable, that is, Hamilton's characteristics function is completely separable in all its variables. The action variables are defined as

$$\boxed{J_i = \oint p_i dq_i}$$

where i is not summed over the cyclic integral is performed over a full periodic variation in q_i.

If the complete integral of the time independent HJ equation,

$$H\left(q_1 \ldots\ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots., \frac{\partial S}{\partial q_n}\right) = E$$

is $$S\left(q_1, \ldots., q_n; \alpha_1, \ldots\ldots, \alpha_n\right) = \sum_{i=1}^{n} S_i\left(q_i; \alpha_1, \ldots\ldots, \alpha_n\right)$$

where one of the $\{\alpha_i\}$ is the energy E, then

$$J_i = \oint p_i dq_i = \oint \frac{\partial S}{\partial q_i} dq_i = \oint \frac{dS_i}{dq_i} dq_i$$

= a function independent of q_i but dependent on $\alpha_1, ..., \alpha_n$ only.

that is, $J_i = J_i(\alpha_1, ..., \alpha_n) \qquad i = 1, ..., n$

Now, these 'n' relations can be inverted to obtain

$$\alpha_i = \alpha_i(J_1, ..., J_n) \quad i = 1, ..., n$$

So, that the complete integral can also be written as

$$\tilde{S}(q_1, ..., q_n; J_1, ..., J_n)$$

instead of $S(q_1, ..., q_n; \alpha_1, ..., \alpha_n)$. $\{J_i\}$ are here the constants of motion (although called action 'variables'!). We can regard them as new momenta P_i for a canonical transforamtion with a new set of coordinates Q_i, conjugate to these momenta $P_i, (i = 1, ..., n)$. This canonical transformation may be generated by a generating function of the type F_2, same as the complete integral of the time independent HJ equation or Hamilton's characteristic function $\tilde{S}(q_1, ..., q_n; J_1, ..., J_n)$. The conditions are

$$Q_i = \frac{\partial \tilde{S}}{\partial P_i} = \frac{\partial \tilde{S}}{\partial J_i} \text{ and } p_i = \frac{\partial \tilde{S}}{\partial q_i}$$

with the new Hamiltonian

$$K(P, Q) = H(p, q) = E(J_1, ..., J_n)$$

which does not depend on $\{Q_i\}$

The canonical equations of motion corresponding to the new hamiltonian K are

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0$$

and

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = \frac{\partial E}{\partial J_i} = \text{constant} = \nu_i = say$$

where $\{\nu_i\}$ is another set of constants. Note that $Q_i \neq 0$ unlike its counterpart in the time dependent HJ equation, where Jacobi's suggested $\{Q_i\}$ to be constants of motion. Here, the solution for $\{Q_i\}$ is

$$Q_i = \nu_i t + \beta_i$$

where $$\nu_i = \frac{\partial E}{\partial J_i} = \text{constant}$$

and β_i are the constants of integration. Again Q_is are the coordinates conjugate to momenta J_is and J_is are of the same dimension as that of angular momentum or action or Plancks' constant $\hbar$. Hence, Q_is are basically angle like variables and are therefore dimensionless. These Q_is are historically denoted by $\omega_i s$ and are called angle variables, given by

$$\omega_i = \nu_i t + \beta_i$$

Obviously, ν_is look like frequencies of oscillation of the coordinate q_i which give rise to J_i, and $\beta_i s$ are the phase constants.

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Thus if the relation $E = E(J_1, ..., J_n)$ is known, the frequencies of oscillation ν_is for all the coordinates given by

$$\nu_i = \frac{\partial E}{\partial J_i}$$

can be obtained. If two or more frequencies of oscillation are identical, such a periodic system is called a degenerate periodic system. If further, all the frequencies $\{\nu_i\}$ are found to be identical, the system is termed as a completely degenerate periodic system.

Now, the new set of variables (ν_i, J_i) should completely specify the dynamical state of any conservative periodic system and

$$\omega_i = \frac{\partial \tilde{S}}{\partial J_i} = \nu_i t + \beta_i$$

can produce the general solution,

$$q_k = q_k(J_1, ..., J_n;\ \omega_1, ..., \omega_n)$$

which can be represented by a multiple Fourier expansion of the fundamental frequencies $\{\nu_i\}$, given by

$$q_k = \sum_{n_1, ... n_N = -\infty}^{\infty} A^{(k)}_{n_1 ..... n_N}(J_1, ..., J_n) \exp\left\{2\pi\sqrt{-1}\sum_{j=1}^{N}(n_j \nu_j t + \beta_j)\right\}$$

where N is the number of DOF of the system, and where the summation in the front extends over all integer values of n_j.

The motion in general is not periodic, its orbit having the characteristics of an 'open' Lissajous figure. Only if the ν_i's are commensurate does the current point of the orbit ever return strictly to its starting point.

The use of this technique was limited to astronomers only, until it was brought into physics by Stacketl, Epstein, and Schowarzschild. If these N frequenices of the system satisfy $n \leq N$ linear equations with integral coefficients, the system is said to have an n-fold degeneracy. In this case, the orbit does not, as in the non-degenerate case, fulfil in the configuration space an N-fold, but only an (N–n)-fold continuum, and the system can be transformed by an appropriate change of variables into a system with (N–n) degrees of freedom.

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1-D Simple Harmonic Oscillator:

The Hamiltonian for such a periodic system is given by

$$H = \frac{1}{2m}p^2 + \frac{1}{2}kq^2$$

which sets the time independent HJ equation as

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}kq^2 = E$$

Therefore, $$J = \oint p dq = \oint \frac{\partial S}{\partial q} dq = \oint \sqrt{2m\left(E - \frac{1}{2}kq^2\right)} dq = 2\sqrt{\frac{m}{k}} E\pi$$

and hence the frequency oscillation is

$$\nu = \frac{\partial E}{\partial J} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

The Kepler Problem:

The motion is periodic if the energy $E < 0$. The Hamiltonian is

$$H = \frac{1}{2\mu}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta} \right) + V(r)$$

Taking $S(r,\theta,\phi) = S_r(r) + S_\theta(\theta) + \alpha_\phi \phi$

the HJ equation reduces to

$$\left(\frac{dS_\theta}{d\theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta} = \alpha_\theta^2 \equiv p_\theta^2 + \frac{\alpha_\phi^2}{\sin^2\theta}$$

and

$$\left(\frac{dS_r}{dr}\right)^2 + \frac{\alpha_\theta^2}{r^2} = 2\mu\{E - V(r)\} \equiv p_r^2 + \frac{\alpha_\theta^2}{r^2}$$

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